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WHEN DO TWO PLANTED GRAPHS HAVE THE SAME COTRANSVERSAL MATROID?

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ABSTRACT. Cotransversal matroids are a family of matroids that arise from planted graphs. We prove that two planted graphs give the same cotransversal matroid if and only if they can be obtained from each other by a series of local moves.

1. Introduction

Cotransversal matroids are a family of matroids that arise from planted graphs. The goal of this short note is to describe when two planted graphs give rise to the same cotransversal matroid.

The paper is organized as follows. In Section 2 we recall some basic definitions and facts in matroid theory, including the notions of cotransversal and transversal matroids. In Sections 3 and 4 we introduce the operations of *swapping* and *saturating* on a planted graph, and prove that they preserve the cotransversal matroid (Theorems (3.2) and (4.2.1)). In Section 5 we prove a crucial lemma on transversal matroids. Finally in Section 6 we prove our main result: two planted graphs give rise to the same cotransversal matroid if and only if their saturations can be obtained from each other by a series of swaps (Theorem (6.1)).

This paper is inspired by an analogous work of Whitney on presentations of graphical matroids. He showed in [10] that two graphs give rise to the same graphical matroid if and only if they can be obtained from each other by repeatedly applying three operations. Our main theorem is also analogous to Bondy [3] and Mason's [5] elegant theorem that a transversal matroid has a unique maximal presentation. In Sections 4 and 5 we will explain how our theorem and theirs are connected by matroid duality, and we will see the need to resolve several subtleties that do not arise in that dual setting.

2. Preliminaries

Matroids can be thought of as a notion of independence, which generalizes various notions of independence occuring in linear algebra, field theory, graph theory, and matching theory, among others. We begin by recalling some basic notions of the theory of matroids. For a more thorough introduction, we refer the reader to [2], [7], [9].

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Definition (2.1). A *matroid* $(E, \mathcal{B})$ consists of a finite set E and a nonempty family $\mathcal{B}$ of subsets of E, called *bases*, with the following property: If $B_a, B_b \in \mathcal{B}$ and $x \in B_a - B_b$, then there exists $y \in B_b - B_a$ such that $(B_a - x) \cup y \in \mathcal{B}$.

A prototypical example of a matroid consists of a finite collection of vectors E spanning a vector space V, and the collection $\mathcal{B}$ of subsets of E which are bases of V.

Matroids have a useful notion of duality, as follows.

Definition (2.2). If $M = (E, \mathcal{B})$ is a matroid then $\mathcal{B}^* = \{E - B \mid B \in \mathcal{B}\}$ is also the collection of bases for a matroid $M^* = (E, \mathcal{B}^*)$, called the *dual* of M.

Note that $(M^*)^* = M$. This allows us to talk about *pairs of dual matroids*.

Duality behaves beautifully with respect to many of the natural concepts on matroids. In particular, the general theory makes it straighforward to translate many notions and results (*e.g.* definitions, constructions, and theorems) about M into "dual" notions and results about M^*.

(2.1) Cotransversal and transversal matroids. We are particularly interested in two families of matroids arising in graph theory and matching theory. First we define *cotransversal matroids*, which are the main object of study of this paper. A vertex of a directed graph G is called a *sink* if it has no outgoing edges. A *routing* is a set of vertex-disjoint directed paths in G.

Definition (2.1.1). A *planted graph* (G, B) is a directed graph G with vertex set V having no loops or parallel edges, together with a specified set of sinks $B \subseteq V$.

Theorem (2.1.2). ([6], [7]). *Given a planted graph (G, B) on V, there is a matroid $L(G, B)$ on V whose bases are the sets of $|B|$ vertices that can be routed to B through vertex-disjoint directed paths.*

Any matroid M that arises in this way is called *cotransversal*, and a planted graph giving rise to it is called a *presentation* of M.

Example (2.1.3). Figure 1 shows a planted graph G with a specified set of sinks, $B = \{4, 5, 6\}$. The bases of the cotransversal matroid $M = L(G, B)$ are all 3-subsets of $\{1, 2, 3, 4, 5, 6\}$ except 245 and 356.

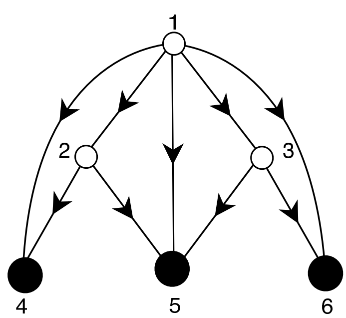

Figure 1. A planted graph (G, B) with $B = \{4, 5, 6\}$.

Now we define *transversal matroids*, another important family.

Definition (2.1.4). Let S be a finite set. Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be a family of subsets of S. A *system of distinct representatives* (*SDR*) of $\mathcal{A}$ is a choice of an element $a_i \in A_i$ for each i such that $a_i \neq a_j$ for $i \neq j$. A *transversal* is a set which can be ordered to obtain an SDR.

THEOREM (2.1.5). ([7]). *Given a family $\mathcal{A} = \{A_1, \dots, A_r\}$ of subsets of S, there is a matroid on S whose bases are the transversals of $\mathcal{A}$.*

A matroid that arises in this way is called a *transversal matroid*, and $\mathcal{A}$ is called a *presentation* of it. We can also view $\mathcal{A} = \{A_1, \dots, A_r\}$ as a bipartite graph between the "top" vertex set $[r] = \{1, \dots, r\}$ and the "bottom" vertex set S, where top vertex i is connected to the elements of A_i for $1 \leq i \leq r$. The SDRs of $\mathcal{A}$ become maximal matchings of $[r]$ into S in this bipartite graph. We will use these two points of view interchangeably.

Example (2.1.6). Let $S = \{1, \dots, 6\}$ and $\mathcal{A} = \{\{1, 2, 3, 4, 5, 6\}, \{2, 4, 5\}, \{3, 5, 6\}\}$. The bases of the resulting transversal matroid M^* are all 3-subsets of $\{1, 2, 3, 4, 5, 6\}$ except 124 and 136.

Note that the cotransversal matroid M of Example (2.1.3) is dual to the transversal matroid M^* of Example (2.1.6). This is a special case of a general phenomenon:

THEOREM (2.1.7). ([1], [4], [7]). *Cotransversal matroids are precisely the duals of transversal matroids.*

Cotransversal matroids were originally called *strict gammoids*. Ingleton and Piff's discovery of Theorem (2.1.7) prompted their newer, widely adopted name.

3. Swapping

In this section we introduce the *swap* operation on planted graphs, and show that it preserves the cotransversal matroid.

In a planted graph, denote the edge from vertex i to vertex j by e_{ij}.

Definition (3.1). Let (G, B) be a planted graph, and let $i \notin B, j \in B$ be such that $e_{ij} \in G$. The *swap operation* **swap(i, j)** turns (G, B) into the planted graph $(G, B)_{i \to j} = (G', B')$ by

- replacing $e_{ij} \in G$ with $e_{ji} \in G'$,
- replacing every other edge of the form e_{ik} in G with $e_{jk} \in G'$, and
- replacing the sink $j \in B$ with the new sink $i \in B'$.

Figure 2 illustrates the operation **swap(i, j)**; the set B is represented by large, black vertices. Note that **swap(j, i)** is a two-sided inverse of **swap(i, j)**.

THEOREM (3.2). *Swaps preserve the cotransversal matroid: If (G, B) is a planted graph, and $i \notin B, j \in B$ are such that $e_{ij} \in G$, then $L((G, B)_{i \to j}) = L(G, B)$.*

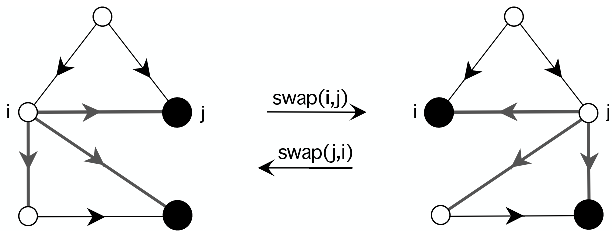

Figure 2. The operation **swap(i, j)**; sinks are drawn as large black vertices.

Proof. Since **swap(i, j)** is invertible, it suffices to show that any set of vertices which could be routed to B in (G, B) can be routed to B' in $(G, B)_{i\to j} = (G', B')$.

Let A be a basis of $L(G, B)$, and consider a routing R from A to B. Let p_{ab} be the path in R which goes from a to b, and let v be the vertex of A which gets routed to j. We consider three cases: (i) v is routed through i to get to j, (ii) v is routed to j without going through i, and i is not in any other route of R, and (iii) v is routed to j without going through i, and i is in some other route of R.

(i) Since e_{ij} is in G, we can assume that R uses the path $p_{vj} = (v, \dots, i, j)$ from v to j. As a result of the operation **swap(i, j)** we have $B' = B - j \cup i$. The operation **swap(i, j)** does not affect the path from v to i, or any other paths in R. We can replace the path p_{vj} in R with the path $p'_{vi} = p_{vj} - e_{ij}$ of G', and let the other paths of the routing stay the same. Therefore A is a basis of $L(G', B')$.

(ii) Since i is not on the route from v to j, no edges along the path p_{vj} are affected by the swap, so v still has this path to j in G'. Also $e_{ji} \in G'$, so the path $p'_{vi} = p_{vj} \cup e_{ji}$ in (G', B') routes v to i and doesn't intersect the other paths of the routing. We obtain that A is a basis of $L(G', B')$.

(iii) Let w be the vertex of A which is routed through i to some sink $b \in B$, $b \neq j$, as shown in Figure 3. As a result of **swap(i, j)**, the path p_{wb} in (G, B) gets blocked at the edge e_{ik}. We can use the truncated path $p'_{wi} = (w, \dots, i)$ in (G', B') as a route from w to $i \in B'$. To complete a routing we need a path leaving $v \in A$ and arriving at $b \in B'$. The path p_{vj} in G is unaffected in G', and $e_{jk} \in G'$ since $e_{ik} \in G$. So we can use the old path p_{vj} and the new edge $e_{jk} \in G'$ to pick up the old path from k to b; this does not intersect any other path in the routing R. It follows that A is a basis of $L(G', B')$. □

4. Saturation for cotransversal matroids

In this section we will see that every presentation (G, B) of a cotransversal matroid $M = L(G, B)$ can be "saturated" in a unique way into a maximal planted graph $\overline{(G, B)} \supseteq (G, B)$ such that $M = L\overline{(G, B)}$. This is done by adding to (G, B) all missing edges that will not affect the cotransversal matroid. This was essentially proved in [3, 5]; to explain it, we need to take a closer look at the duality between cotransversal and transversal matroids.

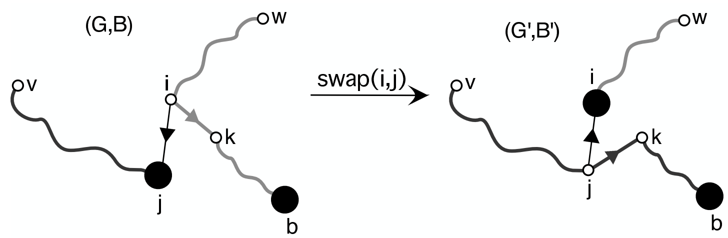

Figure 3. Case (iii): Rerouting v and w.

(4.1) Duality between transversal and cotransversal matroids revisited. In Theorem (2.1.7) we saw that transversal matroids and cotransversal matroids are dual to each other. We will need a slightly stronger version of this statement.

THEOREM (4.1.1). ([4]). *Let M and M^* be a pair of dual cotransversal and transversal matroids on V. Then there is a bijection that maps a planted graph presentation of M to a presentation of M^* together with an SDR.*

The previous theorem is implicit in [4]. For that reason we omit its proof, but we describe the bijection.

Given a planted graph presentation (G, B) of M, let $A_i := \{i\} \cup \{u \mid e_{iu} \in G\}$ for each $i \in V - B$. The sets A_i with $i \in V - B$ make up a presentation of M^*, and the matching of i with A_i is an SDR for those sets.

In the opposite direction, consider a presentation $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of M^* and an SDR $a_1, \ldots, a_k$. For each $x \in A_j$ with $x \neq a_j$, draw the directed edge from a_j to x in G. Let B be the complement of $\{a_1, \ldots, a_k\}$. This will give a presentation of M.

The reader may find it instructive to check that the planted graph presentation of M in Example (2.1.3) is dual to the presentation of M^* in Example (2.1.6) with SDR $(1, 2, 3)$.

(4.2) Saturating a graph. As mentioned in Section 2, theorems about a matroid M can often be translated automatically into "dual" theorems about the dual matroid M^*. This is very useful for our purposes. In their foundational work on transversal matroids, Bondy [3] and Mason [5] explained how the different presentations of a transversal matroid are related to each other. Using Theorem (4.1.1), we will now "dualize" their work, to obtain for free several useful results about the presentations of a cotransversal matroid.

The statements in this section are not difficult to show directly. Since they are dual to results in [3] and [5], we omit their proofs.

THEOREM (4.2.1). ([3], [5]). *For any planted graph (G, B) there exists a unique maximal planted graph $\overline{(G, B)}$ containing (G, B) such that $L\overline{(G, B)} = L(G, B)$. We call $\overline{(G, B)}$ the* saturation *of (G, B).*

Theorem (4.2.1) is all that we need to prove our main result, Theorem (6.1). In the rest of this section, which is logically independent from the remainder of the paper, we describe *how* one constructs the saturation $\overline{(G, B)}$ of (G, B). First we need some definitions.

Definition (4.2.2). Let $M = (E, \mathcal{B})$ be a matroid. Let $K \subseteq E$ and let B_K be a basis of K. The *contraction of M by K*, denoted M/K, is the matroid on $E - K$ whose bases are the sets $B' \subseteq E - K$ such that $B' \cup B_K$ is a basis of M.

It is known ([9] Chapter 5); that any contraction $L(G, B)/K$ of a cotransversal matroid is also cotransversal. To obtain an explicit presentation of it, we first need a presentation (G', B') of $L(G, B)$ with $|K \cap B'| = r(K)$, where $r(K)$ is the maximum number of paths in a routing from K to B in (G, B). To construct it, start with the planted graph (G, B). If $|K \cap B| < r(K)$, there must be a path from some $k \in K$ to some $b \in B - K$. Performing successive swaps on the edges along this path, one obtains a new presentation (G_1, B_1) where $B_1 = B - b \cup k$ satisfies $|K \cap B_1| > |K \cap B|$. By repeating this procedure, we will eventually reach a presentation (G', B') of the matroid with $|K \cap B'| = r(K)$.

Finally, delete from (G', B') the vertices in K and all the edges incident to them. It is easy to check that the resulting planted graph is a presentation of the contraction $L(G, B)/K$.

Definition (4.2.3). Let v be a vertex of a planted graph (G, B). The *claw* of v in (G, B) is $K_v = v \cup \{i \mid e_{vi} \in G\}$.

Recall that a *loop* in a matroid is an element that does not occur in any basis of the matroid. In a cotransversal matroid $L(G, B)$, a loop is a vertex of G from which there is no path to B. The following proposition tells us which edges we can add to (G, B) without changing the cotransversal matroid.

Proposition (4.2.4). ([3], [5]). *Let (G, B) be a planted graph and let v and w be two vertices of G with $v \notin B$. Then $L(G \cup e_{vw}, B) = L(G, B)$ if and only if w is a loop in $L(G, B)/K_v$.*

Therefore, to construct the saturation $\overline{(G, B)}$ of a planted graph (G, B), one successively *saturates each vertex $v \notin B$* as follows: one contracts the matroid by the claw K_v, finds the loops in the resulting planted graph, and connects v to those loops. In Proposition (4.2.4), the condition for adding the edge e_{vw} depends only on the matroid $L(G, B)$ and the claw K_v, neither of which is affected by the saturation of a different vertex $v' \neq v$. It follows that one can saturate the vertices in any order, and one will always end up with the same graph $\overline{(G, B)}$.

5. An exchange lemma for transversal matroids

Theorem (5.1). ([3], [5]). *A transversal matroid has a unique maximal presentation: For every family $\mathcal{A} = \{A_1, \ldots, A_n\}$ of subsets of a set S there is a unique family $\overline{\mathcal{A}} = \{\overline{A_1}, \ldots, \overline{A_n}\}$ of inclusion-maximal subsets of S such that $A_i \subseteq \overline{A_i}$ for $1 \leq i \leq n$, and $\mathcal{A}$ and $\overline{\mathcal{A}}$ give rise to the same transversal matroid.*

The following lemma on SDRs will be crucial later on.

LEMMA (5.2) (SDR exchange lemma). *Suppose that* $\mathcal{A} = \{A_1, \ldots, A_r\}$ *satisfies the* dragon marriage condition:[1] *for all nonempty sets* $\{i_1, \ldots, i_k\} \subseteq [r]$ *we have* $|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \geq k+1$. *Then for any two SDRs M and M′ of* $\mathcal{A}$, *there is a sequence* $M = M_1, \ldots, M_s = M'$ *of SDRs of* $\mathcal{A}$ *such that* M_i *and* M_{i+1} *differ in exactly one position for* $1 \leq i \leq s-1$.

Proof. Construct a graph H in which the vertices are the SDRs of $\mathcal{A}$ and two SDRs are connected by an edge if they differ in only one position. We need to prove that H is connected.

Suppose H is not connected. Consider two SDRs $M_b = (b_1, \ldots, b_r)$ and $M_c = (c_1, \ldots, c_r)$ in distinct components of H. Assume M_b and M_c are chosen so that the Hamming distance $|M_b - M_c|$, *i.e.* the number of positions where M_b and M_c differ, is minimal. We consider the following two cases.

(i) If $\{b_1, \ldots, b_r\} \neq \{c_1, \ldots, c_r\}$, then for some i we have $b_i \notin \{c_1, \ldots, c_r\}$. Then $M_c' = (c_1, \ldots, b_i, \ldots, c_r)$ is an SDR in the connected component of M_c, and satisfies $|M_b - M_c'| < |M_b - M_c|$.

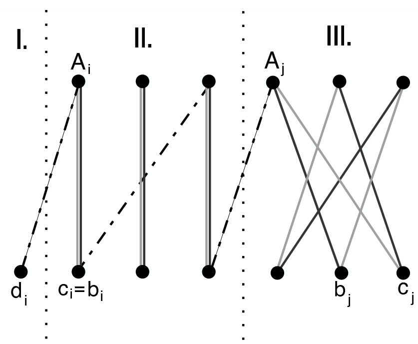

Figure 4. Case (ii): T is partitioned into three parts according to the dark and light SDRs M_b and M_c.

(ii) Suppose $\{b_1, \ldots, b_r\} = \{c_1, \ldots, c_r\}$. We can partition the vertices of our bipartite graph T into three parts based on the matchings M_b and M_c, as shown in Figure 4. (The dotted edges will be explained later.) Part I consists of the vertices of T that are neither in M_b nor in M_c. Part II consists of the top vertices i such that $b_i = c_i$, and the bottom vertices matched to them. Part III consists of the remaining vertices.

The dragon marriage condition gives $|S| \geq r+1$, so there is some $d_i \in A_i$ such that $d_i \notin \{b_1, \ldots, b_r\}$. Therefore $M_b' = (b_1, \ldots, d_i, \ldots, b_r)$ and $M_c' = (c_1, \ldots, d_i, \ldots, c_r)$ are SDRs which are in the connected components of M_b and M_c. We must have $b_i = c_i$, or else $|M_b' - M_c'| < |M_b - M_c|$. In Figure 4, this means that there are no edges from the top of Part III to Part I.

[1]This name is due to Postnikov, and originates as follows. Suppose that S is the set of women and $\{1, \ldots, r\}$ is the set of men in a village, and let A_i be the set of women who are willing to marry man i. A dragon comes to the village and takes one of the women. When is it the case that all the men can still get married, regardless of which woman the dragon takes away? Postnikov showed that this is the case if and only if $\mathcal{A}$ satisfies the dragon marriage condition.

By the dragon marriage condition, the top of Part III must be connected to the bottom of Part II. Define a *zigzag path* to be a path such that:

- its starting point is a vertex in the top of Part III,
- this is the only vertex of Part III it contains, and
- every second edge is a common edge of the matchings M_b and M_c.

We claim that there is at least one zigzag path that ends in Part I. To verify this, consider the set U of vertices in the top that can be reached by a zigzag path starting from the top of Part III. Note that every top vertex in Part III is in U. By the dragon marriage condition, some vertex in U must be connected to a vertex d in the bottom of the graph that is not matched to U in M_b and M_c. If d was in Part II, it would be matched in M_b and M_c to a top vertex $A \notin U$; the edge from d to A would complete a zigzag path that contains A, contradicting our definition of the set U. Therefore d is in Part I.

Consider a zigzag path to d starting at A_j, as shown in Figure 4. Now construct new SDRs M'_b and M'_c by unlinking b_j and c_j from A_j in M_b and M_c respectively, as well as all the edges of M_b and M_c along the zigzag path P. Instead, in both M_b and M_c, rematch the vertices along the edges of path P which were not used by M_b and M_c; these are dotted in Figure 4. Figure 5 shows the resulting new matchings M'_b and M'_c in this example. Now note that

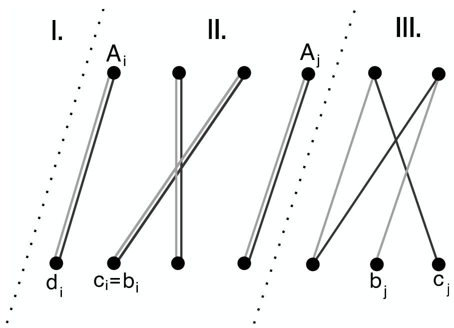

Figure 5. The new matchings M'_b and M'_c.

$|M'_b - M'_c| < |M_b - M_c|$, and M'_b and M'_c are in the same connected components of H as M_b and M_c, respectively. This is a contradiction, and we conclude that H is connected. □

6. The main result

We have now laid all the necessary groundwork to present our main theorem.

THEOREM (6.1). *Two planted graphs (G, B) and (H, C) have the same cotransversal matroid if and only if their saturations $\overline{(G, B)}$ and $\overline{(H, C)}$ can be obtained from each other by a series of swaps.*

Proof. The backward direction follows from Theorems (3.2) and (4.2.1). Now suppose (G, B) and (H, C) are presentations of the same cotransversal matroid M. When we apply the bijection of Theorem (4.1.1) to them, both saturations

$\overline{(G, B)}$ and $\overline{(H, C)}$ must give rise to the unique maximal presentation $\mathcal{A}$ of the dual transversal matroid M^*. They correspond to different matchings M_1 and M_2 of $\mathcal{A}$.

Since $\mathcal{A}$ has at least one matching, we have $|A_{i_1} \cup \cdots \cup A_{i_k}| \geq k$ for all $\{i_1, \ldots, i_k\}$ by Hall's theorem. If we have $|A_{i_1} \cup \cdots \cup A_{i_k}| = k$ for some $\{i_1, \ldots, i_k\}$, then all the elements of $A_{i_1} \cup \cdots \cup A_{i_k}$ are in every basis of M^*. Such elements are called *coloops* of M^* and they correspond to loops in M. By maximality, the loops of M form a complete subgraph in both $\overline{(G, B)}$ and $\overline{(H, C)}$. This is because loops have no path to the sinks; so they cannot be connected to vertices having paths to the sinks, but they can have any possible connection among themselves. We can then restrict our attention to the non-loops of M, where the dragon marriage condition is satisfied.

Applying Lemma (5.2), we can get from M_1 to M_2 by exchanging one element of the matching at a time. One easily checks that these matching exchanges in the bipartite graph correspond exactly to swaps in the corresponding planted graphs under the bijection of Theorem (4.1.1). It follows that one can get from $\overline{(G, B)}$ to $\overline{(H, C)}$ by a series of swaps, as desired. □

We end by illustrating Theorem (6.1) with two examples.

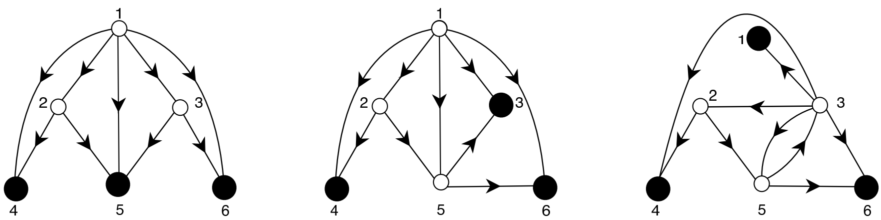

Figure 6. The planted graphs given by $\mathcal{A} = \{\{1, 2, 3, 4, 5, 6\}, \{2, 4, 5\}, \{3, 5, 6\}\}$ with SDRs $(1, 2, 3)$, $(1, 2, 5)$, and $(3, 2, 5)$, respectively.

Example (6.2). Figure 6 shows three saturated planted graph presentations of the cotransversal matroid of Example (2.1.3). They correspond to the dual maximal presentation $\mathcal{A} = \{\{1, 2, 3, 4, 5, 6\}, \{2, 4, 5\}, \{3, 5, 6\}\}$ of the transversal matroid of Example (2.1.6), with SDRs $(1, 2, 3)$, $(1, 2, 5)$, and $(3, 2, 5)$, respectively. Note how one-position exchanges in the SDRs correspond to swaps in the planted graphs.

Example (6.3). Let M be the cotransversal matroid on $\{1, 2, 3, 4, 5\}$ with bases $\{14, 15, 24, 25, 34, 35, 45\}$. Figure 7 shows the graph of saturated planted graph presentations of M, where two planted graphs are joined by an edge labelled ij if they can be obtained from one another by $\mathbf{swap(i, j)}$. There are nine saturated presentations in two isomorphism classes. We have drawn one representative from each isomorphism class; every other saturated presentation is obtained from one of these two planted graphs by relabelling the vertices.

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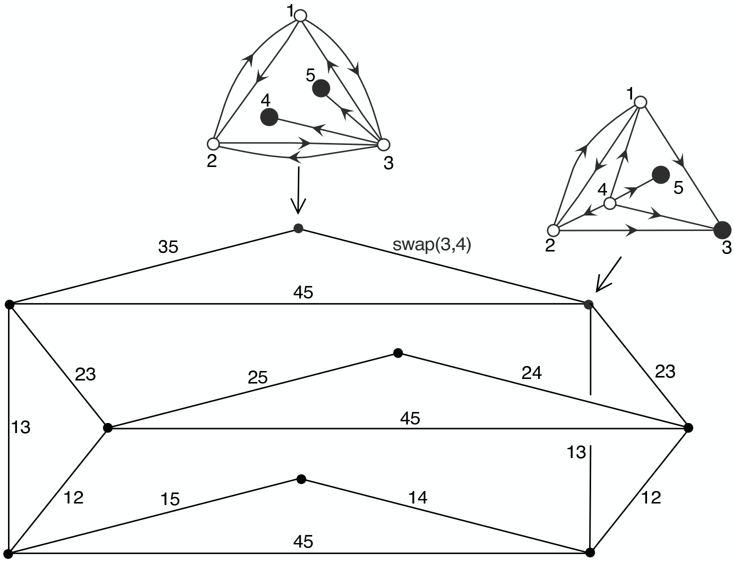

Figure 7. The graph of saturated presentations of a cotransversal matroid.

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ON BOUNDS FOR SOME GRAPH INVARIANTS

ISIDORO GITLER AND CARLOS E. VALENCIA

ABSTRACT. Let G be a graph without isolated vertices, $\alpha(G)$ and $\tau(G)$ be the stability number and the covering number of G, respectively.

The paper is divided in two parts: In the first part we study the minimum number of edges that a k-connected graph can have as a function of $\alpha(G)$ and $\tau(G)$. In particular, we obtain the following lower bound:

$$q(G) \geq \alpha(G) - c(G) + \Gamma(\alpha(G), \tau(G)),$$

where $q(G)$ is the number of edges of G, $c(G)$ is the number of connected components of G and

$$\Gamma(\alpha(G), \tau(G)) = (\alpha(G) - s)\binom{r}{2} + s\binom{r+1}{2},$$

where $\alpha(G) + \tau(G) = r\alpha(G) + s$ with $0 \leq s < \alpha(G)$.

This is a solution to an open question posed by Ore in his book [11], pag. 216; which indeed is a variant for connected graphs of a celebrated theorem of Turán [12].

In the second part of this paper, we study the relations between $\alpha(G)$, $\tau(G)$ and $\delta(G) = \alpha(G) - \sigma_v(G)$, where the *$\sigma_v$-cover* number of a graph, denoted by $\sigma_v(G)$, is the maximum natural number m, such that every vertex of G belongs to a maximal independent set with at least m vertices. The main theorem of this part states that

$$\alpha(G) \leq \tau(G)[1 + \delta(G)].$$

In the last section, we discuss some conjectures related to this theorem.

1. Introduction

Given a graph $G = (V(G), E(G))$, a subset $M \subseteq V(G)$ is a *stable* set if no two vertices in M are adjacent. We say that M is a *maximal stable* set if it is maximal with respect to inclusion. The *stability number* of G is given by

$$\alpha(G) = \max\{|M| \mid M \subset V(G) \text{ is a stable set in } G\}.$$

Also, $C \subseteq V(G)$ is a *vertex cover* for a graph G if every edge of G is incident with at least one vertex in C. Moreover, the vertex cover C is called a *minimal vertex cover* if there is no proper subset of C which is a vertex cover. It is convenient to regard the empty set as a minimal vertex cover for a graph with all its vertices isolated.

The *vertex covering number* of G, denoted by $\tau(G)$, is the number of vertices in a minimum vertex cover in G, that is, the size of any smallest vertex cover in G. Note that a set of vertices in G is a maximal stable set if and only if its complement is a minimal vertex cover for G, thus $\alpha(G) + \tau(G) = |V(G)|$.

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This paper has two main parts. In the first part we solve a problem posed by Ore in his book [11], pag. 216, research prob. 1; whose statement is:

Determine the connected graphs satisfying $\alpha(G) < k$ $(3 \leq k \leq |V(G)|)$ *and having a minimal number of edges.*

This question was completely solved by Turán [12] (see also [11], pages 214–216) when the word connected is removed. In the connected case, we completely solve the question by giving a tight lower bound on the number of edges of a graph of a given order and with given stability number (Theorem (2.2) for connected graphs). This result was independently proved by J. Christophe et al. in [3]. We include the full classification of all optimal graphs (those achieving the bound). A preliminary version of Theorems (2.2) and (3.1) and Lemma (2.8) appear by first time in [5] and [6]. The corresponding result for 2-connected graphs together with a full classification of the optimal graphs is also included (Theorem (2.1.1)). The classification of the connected and 2-connected optimal graphs was obtained independently in [2].

In the second part of the paper we prove the inequality $\alpha(G) \leq \tau(G)(1+\delta(G))$ where $\delta(G) = \alpha(G) - \sigma_v(G)$ and $\sigma_v(G)$ is the largest m, such that every vertex of G belongs to an independent set of size m. As a corollary we obtain $\alpha(G) - |\alpha_{\text{core}}| \leq \tau(G) - |\tau_{\text{core}}|$, where the α_{core} and the τ_{core} are defined as the intersection of all maximum-size stable sets and the intersection of all minimum-size vertex coverings, respectively.

The origin of our interest in the study of these relations comes from monomial algebras. More precisely, the stability number $\alpha(G)$ of a graph G, is equal to the dimension of the Stanley-Reisner ring associated to the graph G, and the covering number $\tau(G)$ of G is equal to the height of the ideal associated to the graph G. Finally, $\sigma_v(G)$ is an upper bound to the depth of this ring.

From the algebraic point of view, an important class of rings is given by those rings R such that their dimension is equal to their depth. The rings in this class are called Cohen-Macaulay rings. A graph is Cohen-Macaulay if the Stanley-Reisner ring associated to it is Cohen-Macaulay. If a graph G is Cohen-Macaulay, then $\delta(G) = 0$ ([14], Proposition 6.1.21). Note that this is a necessary, but not a sufficient condition. The family of graphs with $\delta(G) \geq 1$ corresponds to the Stanley-Reisner rings that have a large depth. Moreover, the dimension minus the depth is bounded below by $\delta(G)$, and hence $\delta(G)$ is a measure of how far these rings are from being Cohen-Macaulay.

The outline of the article is as follows: We begin with Section 2, where we solve the low connectivity (one and two connected) versions of Turán's theorem as thoroughly explained above. In this section we give a lower bound for the number of edges of a graph as a function of its stability and covering numbers (Theorem (2.2)) together with a characterization of q-minimal (Lemma (2.8)) and $\{q, 2\}$-minimal (2-connected) graphs (Theorem (2.1.1)).

In section 3 we study some relations between the stability and covering numbers of a graph. Specifically, we prove the main result (Theorem (3.1)) of this section, which is an inequality that gives an upper bound for the stability number of a graph with respect to the covering number and $\delta(G) = \alpha(G) - \sigma_v(G)$.

This result generalizes an inequality given in [4] which was only valid for B-graphs. Then we introduce the α_{core} and the τ_{core} of a graph and relate them by an inequality with the stability and covering numbers of the graph. Finally, we give a series of conjectures that relate several invariants of graphs for B-graph and hypergraphs.

In this article all graphs are supposed to be finite and simple (i.e., without loops and multiple edges). Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = q$ edges. Given a subset $U \subset V$, the *neighbour set* of U, denoted by $N(U)$, is defined as $N(U) = \{v \in V \mid v \text{ is adjacent to some vertex in } U\}$.

A subset W of V is called a *clique* if any two vertices in W are adjacent. We call W *maximal* if it is maximal with respect to inclusion. The *clique number* of a graph G is given by

$$\omega(G) = \max\{|W| \mid W \subset V(G) \text{ is a clique in } G\}.$$

The *complement* of a graph G, denoted by $\overline{G}$, is the graph with the same vertex set as G, and edges all pairs of distinct vertices that are nonadjacent in G. Clearly, W is a clique of G if and only if W is a stable set of $\overline{G}$, and thus $\omega(G) = \alpha(\overline{G})$.

A subgraph H is called an *induced subgraph* of G, denoted by $G[V(H)]$, if H contains all the edges $\{v_i, v_j\} \in E(G)$ with $v_i, v_j \in V(H)$.

A non-empty graph G is called connected if any two of its vertices are linked by a path in G. A graph G is called *k-connected* (for $k \in \mathbb{N}$) if $|V(G)| > k$ and $G \setminus X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.

2. Low connectivity versions of Turán's theorem

In this section, we study the minimal number of edges in k-connected graphs. Theorem (2.2) establishes a lower bound for the number of edges of a graph G as a function of $\alpha(G)$, $\tau(G)$ and its number of connected components, $c(G)$. As a byproduct of the proof of Theorem (2.2), we find a bound for 2-connected graphs and determine the graphs for which these bounds are sharp.

A *Turán graph*, denoted by $T(a,t)$, is a graph of order $a + t$ consisting of the disjoint union of $a - s$ cliques of order $r = \lfloor \frac{a+t}{a} \rfloor$ and s cliques of order $r + 1$, where $a + t = ra + s$ with $0 \leq s < a$.

For a graph $G = (V, E)$, we denote by $q(G)$ the cardinality of its edge set $E(G)$. We say that a k-connected graph G is *$\{q, k\}$-minimal*, if there is no graph G' such that

(i) G' is k-connected,
(ii) $\alpha(G') = \alpha(G)$,
(iii) $\tau(G') = \tau(G)$, and
(iv) $q(G') < q(G)$.

We say that an edge e of a graph G is an *α-critical edge* if $\alpha(G - e) = \alpha(G) + 1$. A vertex v of a graph G is a *τ-critical vertex* if $\tau(G - v) = \tau(G) - 1$. A connected graph G is called *α-critical* (*τ-critical*) if all its edges (vertices) are α-critical (τ-critical). In Chapter 12 of the book of Lovasz and Plummer [10] some of the basic properties of α-critical graphs can be found. For instance, Corollary 12.1.8 in [10] says that every α-critical graph is 2-connected. Also, by Lemma

12.1.2 in [10], if G is an α-critical graph without isolated vertices, then $\alpha(G) = \alpha(G - v)$ for all $v \in V(G)$. Using the previous observation and the fact that $\alpha(G) + \tau(G) = |V(G)|$ we can conclude that if G is an α-critical graph, then G is a τ-critical graph.

For simplicity a $\{q, 1\}$-minimal graph will be called a q-minimal graph. Hence, if G is q-minimal, then either $\alpha(G) < \alpha(G - e)$ or $c(G) < c(G - e)$ for all the edges e of G (note that $\alpha(G) < \alpha(G - e)$ if and only if $\tau(G) > \tau(G - e)$). That is, an edge of a q-minimal graph is either α-critical or a bridge. Therefore the blocks (a maximal connected subgraph without a cutvertex) of a q-minimal graph are α-critical graphs.

In order to bound the number of edges of a graph we introduce the following numerical function. For any natural numbers a and t, let

$$\Gamma(a, t) = (a - s)\binom{r}{2} + s\binom{r+1}{2},$$

where $a + t = ra + s$ with $0 \leq s < a$. In other words, $r = 1 + \left\lfloor \frac{t}{a} \right\rfloor$ and $s = t - a\left\lfloor \frac{t}{a} \right\rfloor$.

LEMMA (2.1). *Let a and t be natural numbers, then*

(i) $\Gamma(a, t) = \min\left\{\sum_{\mathrm{i}=1}^{\mathrm{a}}\binom{z_i}{2} : \mathrm{z}_1 + \cdots + \mathrm{z}_{\mathrm{a}} = \mathrm{a} + \mathrm{t} \text{ and } z_i \geq 0 \text{ for all } 1 \leq i \leq a\right\}$.

(ii) $\Gamma(a - 1, t) - \Gamma(a, t) \geq \frac{1}{2}(\left\lfloor \frac{t}{a} \right\rfloor^2 - \left\lfloor \frac{t}{a} \right\rfloor) = \binom{\left\lfloor \frac{t}{a} \right\rfloor + 1}{2} \geq 0$ *for all* $a \geq 2$ *and* $t \geq 1$.
Moreover, $\Gamma(a - 1, t) - \Gamma(a, t) = \binom{\left\lfloor \frac{t}{a} \right\rfloor + 1}{2}$ *if and only if* $1 + \left\lfloor \frac{t}{a} \right\rfloor \geq \frac{t}{a-1}$ *and* $\binom{\left\lfloor \frac{t}{a} \right\rfloor + 1}{2} = 0$ *if and only if* $0 \leq t < a$.

(iii) $\Gamma(a, t) - \Gamma(a, t - 1) = 1 + \left\lfloor \frac{t-1}{a} \right\rfloor = \left\lceil \frac{t}{a} \right\rceil$ *for all* $a \geq 1$ *and* $t \geq 2$.

(iv) $\sum_{i=1}^{k} \Gamma(a_i, t_i) \geq \Gamma(\sum_{i=1}^{k} a_i, \sum_{i=1}^{k} t_i)$ *for all* $a_i \geq 1$ *and* $t_i \geq 1$.
Furthermore, if $a_1, a_2 \geq 2$, *then*

$$\Gamma(a_1, t_1) + \Gamma(a_2, t_2) = \Gamma(a_1 + a_2, t_1 + t_2)$$

if and only if either $\left\lfloor \frac{t_1}{a_1} \right\rfloor = \left\lfloor \frac{t_2}{a_2} \right\rfloor$, $\left\lfloor \frac{t_1}{a_1} \right\rfloor - \left\lfloor \frac{t_2}{a_2} \right\rfloor = 1$ *and* $t_1 = r_1 a_1$ *or* $\left\lfloor \frac{t_2}{a_2} \right\rfloor - \left\lfloor \frac{t_1}{a_1} \right\rfloor = 1$ *and* $t_2 = r_2 a_2$.

(v) $\left\lceil \frac{2(a-1+\Gamma(a,t))}{a+t} \right\rceil = 1 + \left\lfloor \frac{t}{a} \right\rfloor + L$, *where* $-1 \leq L \leq 1$.
Moreover, $L = -1$ *if and only if* $a = 1$.

Proof. (i) For $a = 1$ the result is trivial. For $a \geq 2$ we use the next observation: Let $n, m \geq 1$ be natural numbers with $n > m + 1$, then

$$\binom{n}{2} + \binom{m}{2} > \binom{n-1}{2} + \binom{m+1}{2}.$$

Let $a \geq 2$ and $t \geq 1$ be fixed natural numbers, $(z_1, \ldots, z_a) \in \mathbb{N}^a$ such that $\sum_{i=1}^a z_i = a + t$ and let $L(z_1, \ldots, z_a) = \sum_{i=1}^a \binom{z_i}{2}$. Now, if

$$\{z_1, \ldots, z_a\} \neq \{\underbrace{r, \ldots, r}_{a-s}, \underbrace{r+1, \ldots, r+1}_{s}\}$$

where $a+t = ra+s$ with $0 \leq s < a$, then there exist z_{i_1} and z_{i_2} with $z_{i_1} > z_{i_2}+1$. Applying the previous observation we obtain that

$$L(z_1, \ldots z_a) > L(z_1, \ldots, z_{i_1} - 1, \ldots, z_{i_2} + 1, \ldots, z_a) \geq \Gamma(a, t),$$

and therefore we obtain the result.

(ii) Let $a + t = ar + s$ with $0 \leq s < a$, then

$$a + t - 1 = (a - 1)(r + l) + s'$$

where $r + s - 1 = (a - 1)l + s'$ with $l \geq 0$ and $0 \leq s' < a - 1$.

After some algebraic manipulations we obtain that

$$2(\Gamma(a - 1, t) - \Gamma(a, t)) = (r^2 - r) + (l^2 - l)(a - 1) + 2ls'.$$

Therefore $\Gamma(a-1, t)-\Gamma(a, t) \geq \frac{1}{2}(\lfloor \frac{t}{a} \rfloor^2 + \lfloor \frac{t}{a} \rfloor) = \binom{\lfloor \frac{t}{a} \rfloor + 1}{2} \geq 0$, since $r, l, s' \geq 0$ and $u^2 - u \geq 0$ for all $u \geq 0$. Moreover, $\Gamma(a - 1, t) - \Gamma(a, t) = \binom{\lfloor \frac{t}{a} \rfloor + 1}{2}$ if and only if

$$(l, s') = \begin{cases} (0, s') \\ (1, 0) \end{cases}$$

These two possibilities imply that $r + s < a$ and $r + s = a$, respectively. Finally, it is clear that $\binom{\lfloor \frac{t}{a} \rfloor + 1}{2} = 0$ if and only if $0 \leq t < a$.

(iii) Let $a + t - 1 = ar + s$ with $0 \leq s < a$, then

$$a + t = \begin{cases} ar + (s + 1) & \text{if } 0 \leq s < a - 1, \\ a(r + 1) & \text{if } s = a - 1. \end{cases}$$

Hence

$$\Gamma(a, t) - \Gamma(a, t - 1) = \begin{cases} (a-s-1)\binom{r}{2} + (s+1)\binom{r+1}{2} - (a-s)\binom{r}{2} - s\binom{r+1}{2} \\ a\binom{r+1}{2} - \binom{r}{2} - (a - 1)\binom{r+1}{2} \end{cases}$$

$$= \binom{r+1}{2} - \binom{r}{2} = r = \left\lfloor \frac{a + t - 1}{a} \right\rfloor.$$

(iv) Let $a = a_1 + a_2$ and $t = t_1 + t_2$, then by (i)

$$\Gamma(a,t) = \min\left\{ \sum_{i=1}^{a} \binom{z_i}{2} \; : \; \sum_{j=1}^{a} z_j = a+t \text{ and } z_j \geq 0 \text{ for all } 1 \leq j \leq a \right\}$$
$$\leq (a_1 - s_1)\binom{r_1}{2} + s_1\binom{r_1+1}{2} + (a_2 - s_2)\binom{r_2}{2} + s_2\binom{r_2+1}{2}$$
$$= \Gamma(a_1,t_1) + \Gamma(a_2,t_2),$$

where $a_i + t_i = r_i a_i + s_i$ with $0 \leq s_i < a_i$ for all $i = 1, 2$.

In order to have the equality in the previous inequality we need that either $r_1 = r_2$, $r_1 = r_2 + 1$ and $s_1 = 0$ or $r_2 = r_1 + 1$ and $s_2 = 0$.

(v) Let $a + t = ar + s$ with $r \geq 1$ and $0 \leq s < a$. Thus

$$\left\lceil \frac{2(a-1+\Gamma(a,t))}{a+t} \right\rceil = \left\lceil \frac{2\left(a-1+(a-s)\binom{r}{2}+s\binom{r+1}{2}\right)}{a+t} \right\rceil$$
$$= \left\lceil \frac{2(a-1)+r(ar+s)-r(a-s)}{ar+s} \right\rceil$$
$$= r + \left\lceil \frac{2(a-1)-r(a-s)}{ar+s} \right\rceil$$
$$= 1 + \left\lfloor \frac{t}{a} \right\rfloor + \left\lceil \frac{2(a-1)-r(a-s)}{ar+s} \right\rceil = 1 + \left\lfloor \frac{t}{a} \right\rfloor + L.$$

Since $a, r \geq 1$ and $0 \leq s < a$, then $-1 \leq L \leq 1$ because

$$2 < (a+s)(2+r) \Leftrightarrow -2(ar+s) < r(s-a) + 2(a-1)$$
$$\Leftrightarrow -1 \leq L,$$

and

$$2a + rs \leq 2ar + s + 2 \Leftrightarrow 2(a-1) - r(a-s) \leq ar + s$$
$$\Leftrightarrow L \leq 1.$$

Moreover,

$$L = -1 \Leftrightarrow 2(a-1) - r(a-s) \leq -(ar+s) \Leftrightarrow 2a + s(r+1) \leq 2$$
$$\Leftrightarrow a = 1 \text{ and } s = 0.$$

□

THEOREM (2.2) ([5], Theorem 3.3). *Let G be a graph, then*

$$q(G) \geq \alpha(G) - c(G) + \Gamma(\alpha(G), \tau(G)).$$

Proof. We will use induction on $\tau(G)$. The stars $\mathcal{K}_{1,n}$ $(\alpha(\mathcal{K}_{1,n}) = n-1)$ are the unique connected graphs with $\tau(G) = 1$. Since

$$q(\mathcal{K}_{1,n}) = n - 1 = (n-1) - 1 + 1 = \alpha(\mathcal{K}_{1,n}) - c(\mathcal{K}_{1,n}) + \Gamma(n-1, 1),$$

then the result clearly follows. Moreover, the stars $\mathcal{K}_{1,n}$ are q-minimal graphs.

So we can assume that the result is true for $\tau(G) \leq k$ and $k > 1$. Let G be a q-minimal graph with $\tau(G) = k+1$. Now, we will use induction on $\alpha(G)$. If $\alpha(G) = 1$, then G is a complete graph $\mathcal{K}_n$ ($\tau(\mathcal{K}_n) = n-1$). Since

$$q(\mathcal{K}_n) = \binom{n}{2} = 1 - 1 + \binom{n}{2} = \alpha(\mathcal{K}_n) - c(\mathcal{K}_n) + \Gamma(1, n-1),$$

it follows that all the complete graphs satisfy the result.

Hence, we can assume that $\alpha(G) \geq 2$. Furthermore, by Lemma (2.1)(iv), $q(G) = \sum_{i=1}^{s} q(G_i)$, $\alpha(G) = \sum_{i=1}^{s} \alpha(G_i)$ and $\tau(G) = \sum_{i=1}^{s} \tau(G_i)$ where $G_1, \ldots, G_s$ are the connected components of G. Then by the induction hypothesis we can assume that G is a connected graph.

Let e be an edge of G and consider the graph $G' = G - e$. We have two possibilities

$$\tau(G') = \begin{cases} \tau(G) \\ \tau(G) - 1 \end{cases}$$

That is, an edge of G is either a bridge or critical.

Case 1. First, assume that G has no bridges, that is, G is an α-critical graph.

Claim (2.3). Let v be a vertex of G of maximum degree, then

$$\deg(v) \geq 1 + \left\lfloor \frac{\tau(G) - 1}{\alpha(G)} \right\rfloor .$$

Proof. Since any α-critical graph is τ-critical, then $\tau(G - v) = \tau(G) - 1$ and $\alpha(G - v) = \alpha(G)$. Moreover, since the α-critical graphs are 2-connected, then $G - v$ is connected. Now, by the induction hypothesis we have that

$$q(G - v) \geq \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G) - 1).$$

Using the formula

$$\sum_{v_i \in V(G-v)} \deg(v_i) = 2q(G - v)$$

and Lemma (2.1)(v), we conclude that there must exist a vertex $v' \in V(G - v)$ with

$$\begin{aligned} \deg(v') &\geq \left\lceil \frac{2q(G-v)}{|V(G-v)|} \right\rceil \geq \left\lceil \frac{2(\alpha(G) - 1 + \Gamma(\alpha(G), \tau(G) - 1))}{n-1} \right\rceil \\ &\overset{(v)}{\geq} 1 + \left\lfloor \frac{\tau(G) - 1}{\alpha(G)} \right\rfloor + L \geq 1 + \left\lfloor \frac{\tau(G) - 1}{\alpha(G)} \right\rfloor . \end{aligned} \tag{2.4}$$ □

By Lemma (2.1)(iii)

$$\begin{aligned} q(G) = q(G - v) + \deg(v) &\geq \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G) - 1) + \deg(v') \\ &\overset{(2.4)}{\geq} \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G) - 1) + 1 + \left\lfloor \frac{\tau(G) - 1}{\alpha(G)} \right\rfloor \\ &\overset{(iii)}{=} \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G)). \end{aligned} \tag{2.5}$$

So, if the graph G has an edge that is a bridge, then $c(G') = c(G) + 1 = 2$. Let us denote by G_1 and G_2 the connected components of $G - e$. We need to considerer another two cases:

Case 2. Assume that $\tau(G_1) > 0$ or $\tau(G_2) > 0$, then $\tau(G_1) \leq k$, $\tau(G_2) \leq k$, and by the induction hypothesis

$$q(G_1) \geq \alpha(G_1) - 1 + \Gamma(\alpha(G_1), \tau(G_1)) \text{ and } q(G_2) \geq \alpha(G_2) - 1 + \Gamma(\alpha(G_2), \tau(G_2)).$$

Using the above formulas, the fact that $\alpha(G) = \alpha(G_1) + \alpha(G_2)$ and $\tau(G) = \tau(G_1) + \tau(G_2)$, and Lemma (2.1)(iv), we get

$$\begin{aligned} q(G) &= q(G_1) + q(G_2) + 1 \\ &\geq \alpha(G_1) - 1 + \alpha(G_2) - 1 + \Gamma(\alpha(G_1), \tau(G_1)) + \Gamma(\alpha(G_2), \tau(G_2)) + 1 \\ &= \alpha(G) - 1 + \Gamma(\alpha(G_1), \tau(G_1)) + \Gamma(\alpha(G_2), \tau(G_2)) \\ &\overset{\text{(iv)}}{\geq} \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G)). \end{aligned}$$

Case 3. Assume that there does not exist a bridge satisfying the above conditions: for all the bridges of G we have that $\tau(G_1) = 0$ or $\tau(G_2) = 0$. That is, each bridge connects an isolated vertex with the rest of G. In this case, we must have that G is equal to an α-critical graph G_1 with a vertex of G_1 being the center of a star $\mathcal{K}_{1,l}$. Moreover, $\tau(G) = \tau(G_1)$ and $\alpha(G) = l + \alpha(G_1)$ because G_1 is vertex-critical and therefore each vertex belongs to a minimum vertex cover. Using Case 1 and Lemma (2.1)(ii) we obtain

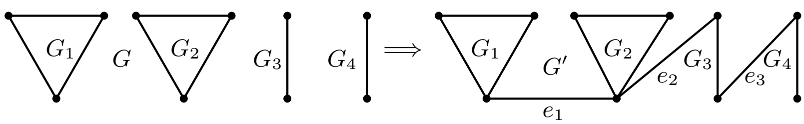
$$\begin{aligned} q(G) = l + q(G_1) &\geq l + \alpha(G_1) - 1 + \Gamma(\alpha(G_1), \tau(G_1)) \\ &= \alpha(G) - 1 + \Gamma(\alpha(G_1), \tau(G)) \overset{\text{(ii)}}{\geq} \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G)). \end{aligned}$$

□

The k-connexion of a graph. A *k-connexion* of a graph G is a k-connected graph G' on the same vertex set as G, with the minimum possible number of edges, and such that G is a subgraph of G'. The graph G is called the *subjacent* graph of the k-connexion graph G' and the edges of G' that are not edges of G are called the *connexion edges*.

Clearly a 1-connexion graph G' of a disconnected graph G can be obtained by adding $c(G) - 1$ edges, where $c(G)$ is the number of connected components of G. This definition is equivalent to the one given in [2] of a *tree-linking* of a graph. In fact, 2-connexions as defined here, are equivalent to *cycle-linkings* as defined in [2].

Example (2.6). In order to illustrate the concept of 1-connexion consider the following graphs:



G' is the 1-connexion of G, the edges e_1, e_2, e_3 are the 1-connexion edges of G', and G_1, G_2, G_4 are the leaves of G'.

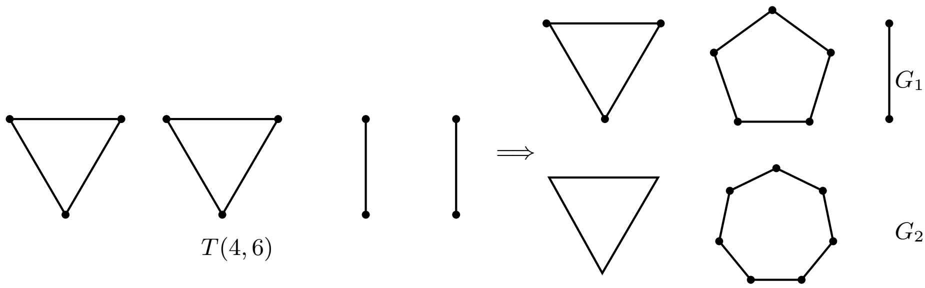
A *leaf* of a 1-connexion G' of a graph G is a connected component G_i of G with the property that there exists a unique vertex v of G_i, such that all connexion edges with one end in G_i are incident to v. If G is a connected graph, then we say that G is a *leaf* of G'. Note that a 1-connexion G' of a graph G has at least one leaf.

Polygon transformed Turán graph. A graph G with covering number $\tau(G) = t$ and stability number $\alpha(G) = a$ is said to be a *polygon transformed Turán graph* or *PTT graph* if either G is isomorphic to $T(a,t)$, or $a \le t < 2a$ and G can be obtained from $T(a,t)$ by the following construction:

Let k_2 and k_3 be the number of copies of K_2 and K_3 in $T(a,t)$ respectively. Let k be a positive integer with $k \le \min\{\mathrm{k}_2, \mathrm{k}_3\}$ and for all $1 \le i \le k$ take positive integers j_i such that $j_1 + \cdots + j_k \le k_2$. Finally, for all $i = 1, \ldots, k$ replace one copy of K_3 and j_i copies of K_2 by a cycle C_{2j_i+3}.

In this way a PTT graph is the disjoint union of complete graphs and possibly odd cycles.

Example (2.7). In order to illustrate the previous concept consider the following graphs:



in the left side it can be seen the Turán graph $T(4,6)$ and in the right side there are two of the three possible polygon transformed graph of $T(4,6)$. Note that G_1 and G_2 are obtained when we take $1 = k < \min\{\mathrm{k}_2, \mathrm{k}_3\} = 2$, and $j_1 = 1$ and $j_1 = 2$ respectively.

LEMMA (2.8). *A graph G is q-minimal if and only if G is a 1-connexion of a polygon transformed Turán graph.*

Proof. ($\Leftarrow$) Let H be a PTT graph with $H_1, \ldots, H_a$ connected components and let L be a 1-connexion of H. Since $\mathcal{K}_r$ and C_{2s+1} are q-minimal, $\alpha(L) = \sum_{i=1}^{a} \alpha(H_i)$, $\alpha(C_{2s+1}) = s$, $q(L) = a - 1 + \sum_{i=1}^{a} q(H_i)$, then L is q-minimal.

($\Rightarrow$) We use double induction on the stability and covering numbers of the graph. For $\alpha(G) = 1$, G must be a complete graph and the result is clear. □

Let G be a q-minimal graph with $\alpha(G) \ge 2$. We can assume that G is an α-critical graph, since if G is not α-critical, then using the arguments used in cases 2 and 3 (in the proof of Theorem (2.2)) and the induction hypothesis, the result follows readily. Since connexion edges of a 1-connexion of a disconnected PTT graph are not α-critical edges ($\alpha(L) = \sum_{i=1}^{k} \alpha(H_i)$, where L is a 1-connexion of a disconnected PTT graph H with connected components $H_1, \ldots, H_k$), then the result follows if we prove that G is either an odd cycle or a complete graph.

Claim (2.9). If v is a vertex of G of maximal degree, then $G - v$ is q-minimal.

Proof. Assume that $G - v$ is not q-minimal. Then, by Claim (2.3) and Lemma (2.1),

$$q(G) = q(G - v) + \deg(v) \overset{(v)}{\geq} \alpha(G) + \Gamma(\alpha(G), \tau(G) - 1) + \left\lfloor \frac{\alpha(G) + \tau(G) - 1}{\alpha(G)} \right\rfloor$$

$$\overset{(iii)}{=} \alpha(G) + \Gamma(\alpha(G), \tau(G)), \tag{2.10}$$

which is a contradiction to the q-minimality of G. □

Since $G - v$ is q-minimal, by the induction hypothesis $G - v$ is a 1-connexion of a PTT graph. Moreover, since G is α-critical, then $G \setminus N[v]$ (where $N[v] = N(v) \cup \{v\}$) is a maximal induced subgraph of G with $\alpha(G \setminus N[v]) = \alpha(G) - 1$. Therefore, we need to determine the maximal induced subgraphs L' of a 1-connexion of a PTT graph L with $\alpha(L') = \alpha(L) - 1$.

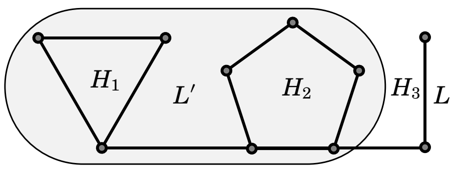
Claim (2.11). Let H be a PTT graph with $H_1, \ldots, H_a$ connected components and L be a 1-connexion of H. If L' is a maximal induced subgraph of L with $\alpha(L') = \alpha(L) - 1$, then

(i) L' is induced by the set of vertices $V(L) \setminus V(H_i)$, for some H_i with $\alpha(H_i) = 1$, or

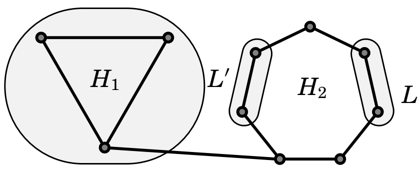
(ii) L' is induced by the set of vertices in $V(L) \setminus \{v_1, v_2, v_3\}$, where $\{v_1, v_2, v_3\}$ are vertices of an odd cycle H_j such that $H_j \setminus \{v_1, v_2, v_3\}$ is a disjoint union of paths with an even number of vertices, or

(iii) L' satisfies the following conditions: (*1*) $V(H_i) \cap V(L') \neq \emptyset$ for all H_i, (*2*) if H_i is an odd cycle, then $V(H_i) \subset V(L')$, (*3*) if H_i is a complete graph such that $V(H_i) \not\subset V(L')$, then for all $v \in V(H_i) \cap V(L')$ there exists at least one connexion edge e_v of L incident to v.

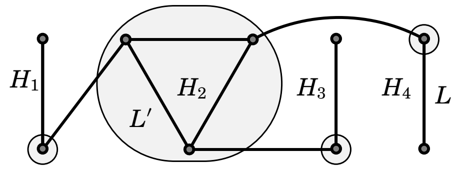
Proof. If $V(L') \cap V(H_i) = \emptyset$ for some $1 \leq i \leq a$ with $\alpha(H_i) = 1$, then $L' = L[V(L) \setminus V(H_i)]$, since $V(L') \subseteq V(L) \setminus V(H_i)$ and $\alpha(L[V(L) \setminus V(H_i)]) = \alpha(L) - 1$.



Therefore we can assume that $V(H_i) \cap V(L') \neq \emptyset$ for all H_i with $\alpha(H_i) = 1$. Let us assume that $V(H_j) \not\subset V(L')$ for some $H_j = C_{2m+1}$. Since all the proper induced graphs of a cycle are paths P_n with $\alpha(P_n) = \lceil \frac{n}{2} \rceil$, then $\alpha(H_j \setminus C) = \alpha(H_j) - 1 = m - 1$ for some $C \subset V(H_j)$ if and only if $H_j[V(H_j) \setminus C]$ is a disjoint union of three paths $P_{m_1}, P_{m_2}, P_{m_3}$ for some even numbers $m_1, m_2, m_3 \geq 0$ such that $m_1 + m_2 + m_3 = 2(m - 1)$. Since L' is a maximal induced subgraph of L with $\alpha(L') = \alpha(L) - 1$, then $V(H_j) \not\subset V(L')$ for only one $H_j = C_{2m+1}$, therefore L' is given by (*ii*).



In order to conclude the proof we can assume that $V(L') \cap V(H_i) \neq \emptyset$ for all $1 \leq i \leq a$ and $V(H_j) \subset V(L')$ for all H_j with $\alpha(H_j) \geq 2$. Clearly, if $v \in V(H_i) \cap V(L')$ such that v is not incident to any connexion edge of L, then $V(H_i) \subset V(L')$ because $\alpha(L') = \alpha(L[V(L') \cup V(H_i)])$.



□

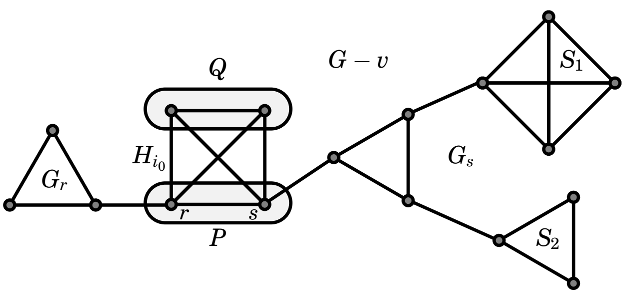
Take $L = G \setminus v$ and $L' = G \setminus N[v]$. If $G \setminus N[v]$ satisfies (i), then G must be a complete graph. If $G \setminus N[v]$ satisfies (ii), then G must be an odd subdivision of the complete graph $\mathcal{K}_4$ and it fails in one edge in order to be a q-minimal graph.

Finally, assume that $G \setminus N[v]$ satisfies (iii). Let H_{i_0} be a connected component of H such that H_{i_0} is a complete graph and $V(H_{i_0}) \not\subset V(G \setminus N[v])$. Take $P = V(H_{i_0}) \cap V(G \setminus N[v])$ and $Q = V(H_{i_0}) \setminus P$. Since $G - v$ is q-minimal, then for all $u \in P$, the graph $(G - v) \setminus u$ is disconnected. For all $u \in P$, let G_u be the disjoint union of the connected components $C_1, \ldots, C_s$ (C_i is a 1-connexion of some PTT graph) of $(G - v) \setminus u$ with $V(C_j) \cap V(H_{i_0}) = \emptyset$. Note that, G_u is an induced subgraph of $G - v$ and its connected components are joined to u by some connexion edges.

Let C be a connected component of G_u and S be a leaf of C not joined to u by a connexion edge. Since G is a 2-connected graph, then v must be incident with at least one vertex of S. If G_u is either a complete graph or an odd cycle, then v must be incident with at least one vertex of G_u not incident with a connexion edge of $G - v$. Moreover, if v_s is the unique vertex of a leaf S of C such that all the connexion edges with one end in S are incident to v_s, then by Claim (2.11) (iii), v must be incident with all the vertices of $S \setminus v_s$. Since v is incident with all the vertices of Q, then

$$\deg(v) \geq |Q| + \sum_{u \in P} \sum_{H_j \in L(G_u)} (|H_j| - 1) \overset{(*)}{\geq} |H_{i_0}|, \tag{2.12}$$

where $L(G_u)$ is either the set of leaves of G_u not joined to u or if G_u is a 2-connected graph, then $L(G_u)$ is $\{G_u\}$. Furthermore, $(*)$ is an equality if and only if G_u is connected, all the leaves of G_u are isomorphic to $\mathcal{K}_2$, and if G_u has at most two leaves.



Using the first inequality in the equation (2.12), it is not difficult to prove that

$$\deg(v) \geq s+(k-s)(k-1) \geq 2k-2, \text{ where } k = \left\lfloor \frac{|V(G-v)|}{\alpha(G)} \right\rfloor \leq |V(H_{i_0})|. \tag{2.13}$$

On the other hand, since G and $G-v$ are q-minimal graphs, then

$$\begin{aligned}\deg(v) &= q(G) - q(G-v) = \Gamma(\alpha(G), \tau(G)) - \Gamma(\alpha(G), \tau(G)-1) \\ &= \left\lfloor \frac{|V(G-v)|}{\alpha(G)} \right\rfloor = \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil - 1 \leq k.\end{aligned}$$

Therefore, $k=2$, $H_{i_0} = \mathcal{K}_2$, $\deg(v) = 2$, $\left\lceil \frac{|V(G-v)|}{\alpha(G)} \right\rceil \leq 3$, $|P| = 1$, and G_u has only two leaves, that is, G is an odd cycle. □

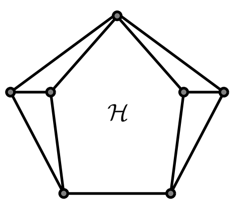
(2.1) The 2-connected case.

THEOREM (2.1.1). *Let G be a 2-connected graph with $\tau(G) \geq 2$, then*

$$q(G) \geq \begin{cases} 2\alpha(G) & \text{if } \tau(G) \leq \alpha(G), \\ \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G)) & \text{if } \alpha(G) = 1 \text{ or } \tau(G) - \alpha(G) = 1, \\ \alpha(G) + \Gamma(\alpha(G), \tau(G)) & \text{otherwise.} \end{cases}$$

Furthermore, G is $\{q, 2\}$-minimal with $\tau(G) \geq \alpha(G)$ if and only if one of the following conditions is satisfied:

(i) *G is an even cycle,*
(ii) *G is the complete graph with at least three vertices,*
(iii) *G is an odd cycle,*
(iv) *G is a 2-connexion of a polygon transformed Turán graph,*
(v) *G is an odd subdivision of the complete graph $\mathcal{K}_4$,*
(vi) *G is isomorphic to the following graph:*



Proof. Let H be a 2-connected graph and G be a $\{q, 2\}$-minimal graph. We will divide the proof in three cases:

Case 1 ($\tau(G) = \alpha(G) > 1$). Let H with $\tau(H) = \alpha(H) > 1$, then $\deg(v) \geq 2$ for all $v \in V(H)$. Therefore

$$q(H) = \frac{\sum_{v \in V(H)} \deg(v)}{2} \geq |V(H)| = 2\alpha(H).$$

Since the even cycle C_{2a} is a 2-connected graph with $\alpha(C_{2a}) = \tau(C_{2a})$, then $2\alpha(G) \leq q(G) \leq 2\alpha(G)$ and $\deg(v) = 2$ for all $v \in V(G)$. Furthermore, since a graph H with all its vertices of degree two is a disjoint union of cycles, then G is a $\{q, 2\}$-minimal graph with $\alpha(G) = \tau(G)$ if and only if G is an even cycle.

Case 2 ($\tau(G) < \alpha(G)$ and $\alpha(G) > 1$). Since $\deg(v) \geq 2$ for all $v \in V(H)$ (H is 2-connected), then $q(H) \geq 2\alpha(H)$. Let $s(\mathcal{K}_{a,t})$ be an odd subdivision of the complete bipartite graph $\mathcal{K}_{a,t}$. Since $s(\mathcal{K}_{a,t})$ is a 2-connected graph with $\alpha(s(\mathcal{K}_{a,t})) = \tau(s(\mathcal{K}_{a,t})) + (a-t)$ and $q(\mathcal{K}_{a,t}) = 2\alpha(\mathcal{K}_{a,t})$, then $q(G) = 2\alpha(G)$ for all the $\{q,2\}$-minimal graphs G with $\tau(G) < \alpha(G) > 1$.

This finishes the proof of the lower bound for the number of edges whenever $\tau(G) \leq \alpha(G)$ and $\alpha(G) > 1$ and the characterization of the $\{q,2\}$-minimal graphs (case (i)) whenever $\alpha(G) = \tau(G)$ and $\alpha(G) > 1$.

Case 3 ($\alpha(G) = 1$ or $\tau(G) \geq \alpha(G) + 1$). We use double induction on the stability and covering numbers of the graph.

If $\alpha(G) = 1$, then G is a complete graph and clearly G is a $\{q,2\}$-minimal graph. If $\tau(H) - \alpha(H) = 1$, then by Lemma (2.8), $q(H) \geq \alpha(H) - 1 + \Gamma(\alpha(H), \tau(H))$ and H is a q-minimal graph with $\tau(H) - \alpha(H) = 1$ if and only if H is an odd cycle. Since the odd cycles are 2-connected, then $q(H) \geq \alpha(H) - 1 + \Gamma(\alpha(H), \tau(H))$ whenever $\tau(H) = \alpha(H) + 1$ and G is $\{q,2\}$-minimal with $\tau(G) = \alpha(G) + 1$ if and only if G is an odd cycle.

By Lemma (2.8) we have that $q(H) \geq \alpha(H) - 1 + \Gamma(\alpha(H), \tau(H))$ whenever H is a connected graph with $\tau(H) > \alpha(H) + 1 > 2$. On the other hand, it is not difficult to see that if H is a graph as in (iv), (v) or (vi), then H is a 2-connected graph with $q(H) = \alpha(H) + \Gamma(\alpha(H), \tau(H))$. Furthermore, since all that q-minimal graphs with $\tau(H) > \alpha(H) + 1 > 2$ are not 2-connected, then $q(H) \geq \alpha(H) + \Gamma(\alpha(H), \tau(H))$ whenever H is a 2-connected graph with $\tau(H) > \alpha(H) + 1 > 2$.

Therefore, to finish the proof we only need to show that G is a $\{q,2\}$-minimal graph with $\tau(G) \geq \alpha(G) + 2$ if and only if G is as in (iv), (v) or (vi). In order to do so, we follow the same sequence of arguments as in the proof of Lemma (2.8).

Let $e \in E(G)$, if e is not an α-critical edge of G, then $G \setminus e$ is a q-minimal graph and G is 2-connexion of a PTT graph. Therefore we can assume that G is an α-critical graph.

Claim (2.1.2). If v is a vertex of G of maximal degree, then either $G - v$ is q-minimal or $\{q,2\}$-minimal.

Proof. If $G \setminus v$ is neither q-minimal nor $\{q,2\}$-minimal, then $q(G \setminus v) \geq \alpha(G) + \Gamma(\alpha(G), \tau(G) - 1) + 1$. Therefore, the result follows using the same arguments that in Claim (2.9). □

Since $G - v$ is either a q-minimal or a $\{q,2\}$-minimal graph, then by the induction hypothesis

$$q(G-v) = \begin{cases} \alpha(G-v) - 1 + \Gamma(\alpha(G-v), \tau(G-v)) & \text{if } G \text{ is } q\text{-minimal,} \\ \alpha(G-v) + \Gamma(\alpha(G-v), \tau(G-v)) & \text{if } G \text{ is } \{q,2\}\text{-minimal.} \end{cases}$$

Since $\deg(v) = q(G) - q(G-v)$ and $\Gamma(\alpha(G), \tau(G)) - \Gamma(\alpha(G), \tau(G)-1) = \left\lfloor \frac{|V(G)|-1}{\alpha(G)} \right\rfloor$ (Lemma (2.1) (*iii*)), then

$$\deg(v) = \begin{cases} \left\lfloor \frac{|V(G)|-1}{\alpha(G)} \right\rfloor + 1 & \text{if } G \text{ is } q\text{-minimal,} \\ \left\lfloor \frac{|V(G)|-1}{\alpha(G)} \right\rfloor & \text{if } G \text{ is } \{q,2\}\text{-minimal.} \end{cases} \tag{2.14}$$
□

Claim (2.1.3). Let H be a PTT graph with $H_1, \ldots, H_a$ connected components and let L be a 2-connexion of H. If L' is a maximal induced subgraph of L with $\alpha(L') = \alpha(L) - 1$, then L' is given as in (i), (ii), and (iii) in Claim (2.11).

Proof. Let e be a connexion edge of L. Since L is a $\{q, 2\}$-minimal graph, then $L \setminus e$ is a 1-connexion of H. Hence, applying Claim (2.11) to $L \setminus e$ we get the result. □

Now we will consider the cases when $G - v$ is either q-minimal or $\{q, 2\}$-minimal:

Case ($G - v$ is q-minimal). Take $L = G \setminus v$ and $L' = G \setminus N[v]$. If $G \setminus N[v]$ is as in Claim (2.11) (i), then G must be a complete graph. If $G \setminus N[v]$ is as in Claim (2.11) (ii), then G must be an odd subdivision of the complete graph $\mathcal{K}_4$.

Now assume that $G \setminus N[v]$ is as in Claim (2.11) (iii). Using equations (2.12) and (2.14), we get that $k + 1 = \deg(v) \geq 2k - 2$, where $k = \lfloor \frac{|V(G)|-1}{\alpha(G)} \rfloor$, that is, $k \leq 3$. If $k = 2$, then G is either an odd cycle or an odd subdivision of $\mathcal{K}_4$ and if $k = 3$, then G is $\mathcal{H}$.

Case ($G - v$ is $\{q, 2\}$-minimal). Take $L = G \setminus v$ and $L' = G \setminus N[v]$. If L' is as in Claim (2.1.3) (i), then G is a 2-connexion of a PTT graph, but it is not an α-critical graph. If $G \setminus N[v]$ is as in Claim (2.11) (ii), then G is an odd subdivision of $\mathcal{K}_4$.

Now assume that $G \setminus N[v]$ is as in Claim (2.1.3) (iii). Using equations (2.12) and (2.14), we get that $k = \deg(v) \geq 2k - 2$, where $k = \lfloor \frac{|V(G)|-1}{\alpha(G)} \rfloor$, that is, $k \leq 2$. If $k = 2$, then G is an odd cycle.

Finally, $G - v$ is not an odd subdivision of $\mathcal{K}_4$ because $\deg(v) \overset{(2.14)}{=} 2$ and if $\mathcal{O}$ is an odd subdivision of $\mathcal{K}_4$, then $\alpha(\mathcal{O} \setminus \{a, b\}) = \alpha(\mathcal{O})$ for all $a, b \in V(\mathcal{O})$. If $G - v$ is $\mathcal{H}$, then $\alpha(\mathcal{H}) = 2$, $\omega(\mathcal{H}) = 3$, and $\deg(v) \overset{(2.14)}{=} 3 \geq |V(G)| - \omega(\mathcal{H}) = 4$; a contradiction.

Remark (2.1.4). After this paper was first submitted in 2006, the authors realized that Theorem 2.2 was also obtained independently in [3].

Remark (2.1.5). It can be proved that for any fixed $\delta_-(G) = \alpha(G) - \tau(G) = k > 0$ there exist a finite number of "basic" graphs such that if G is $\{q, 2\}$-minimal graph with $\delta_-(G) = k$, then G is an odd subdivision of some of this basic graphs. For instance, if G is a $\{q, 2\}$-minimal graph with $\delta_-(G) = 1$, then G is an odd subdivision of the complete bipartite graph $\mathcal{K}_{2,3}$.

3. Some bounds for the stability and covering number of a graph

The following results are in the spirit of [4], where the authors were motivated in bounding invariants for edge rings. In this paper, we concentrate mainly on the combinatorial aspects of these bounds.

The theorem below gives an idea of the class of graphs that are Cohen-Macaulay and of those graphs that are far from being Cohen-Macaulay. We thank N. Alon (private communication) for some useful suggestions for making the proof of this result simpler and more readable.

THEOREM (3.1). *Let G be a graph without isolated vertices, then*

$$\alpha(G) \leq \tau(G)[1 + \delta(G)].$$

Proof. First, let fix a minimal vertex cover C with $\tau(G)$ vertices. Label the vertices of C from 1 to $\tau(G)$. For each $i \in C$, let T_i be a maximal stable set containing i, with $|T_i| \geq \sigma_v(G)$. Let k be the minimal natural number such that

$$C \subseteq \bigcup_{i=1}^{k} T_i.$$

Clearly $0 < k \leq \tau(G)$. Let $M = V(G) \setminus C$ and take $C_i = C \cap T_i$ and $M_i = M \cap T_i$ for all $i = 1, \dots, \tau(G)$. Since M is a maximal stable set and G does not have isolated vertices, then for each vertex $v \in M$ there is an edge $e = \{v, v'\}$ with $v' \in C$. That is,

$$M = \bigcup_{i=1}^{k} (M \cap N(C_i)). \tag{3.2}$$

Since $S_i = V(G) \setminus T_i = (C \setminus C_i) \cup (M \setminus M_i)$ is a minimal vertex cover with $|S_i| \leq n - \sigma_v(G)$ for all $i = 1, \dots, k$, then

$$|C \setminus C_i| + |M \setminus M_i| = |(C \setminus C_i) \cup (M \setminus M_i)| = |S_i| \leq n - \sigma_v(G).$$

Hence, as $M \cap N(C_i) = M \setminus M_i$, then

$$\begin{aligned} |M \cap N(C_i)| = |M \setminus M_i| &\leq n - \sigma_v(G) - |C \setminus C_i| \\ &= |C| + |M| - \sigma_v(G) - |C \setminus C_i| \\ &= |C_i| + \alpha(G) - \sigma_v(G) = |C_i| + \delta(G). \end{aligned} \tag{3.3}$$

Taking

$$A_i = C_i \setminus (\bigcup_{j=1}^{i-1} C_j) \text{ and } B_i = (M \cap N(C_i)) \setminus (\bigcup_{j=1}^{i-1} M \cap N(C_j)),$$

we have that

$$|C_i \setminus A_i| \leq |M \cap N(C_i \setminus A_i)|. \tag{3.4}$$

Indeed, if $|C_i \setminus A_i| > |M \cap N(C_i \setminus A_i)|$, then $C \setminus (C_i \setminus A_i) \cup (M \cap N(C_i \setminus A_i))$ would be a vertex cover of cardinality $|C \setminus (C_i \setminus A_i)| + |M \cap N(C \setminus A_i)| < |C|$; a contradiction.

To finish the proof, we use the inequalities (3.3) and (3.4) to conclude that

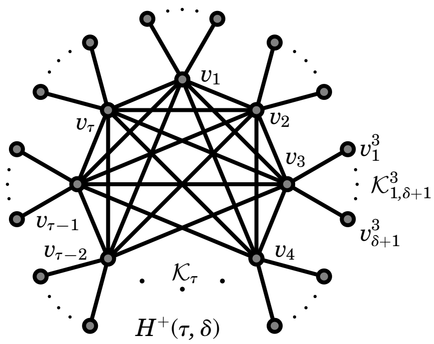
$$\begin{aligned} |B_i| &= |M \cap N(C_i)| - |(M \cap N(C_i)) \cap (\bigcup_{j=1}^{i-1} (M \cap N(C_j)))| \\ &= |M \cap N(C_i)| - |M \cap N(C_i) \cap N(\bigcup_{j=1}^{i-1} C_j)| \\ &\overset{(3.3)}{\leq} |C_i| + \delta(G) - |M \cap N(C_i \cap \bigcup_{j=1}^{i-1} C_j)| \\ &\overset{(3.4)}{\leq} |C_i| + \delta(G) - |C_i \setminus A_i| = |A_i| + \delta(G). \end{aligned} \tag{3.5}$$

Therefore

$$\alpha(G) = |M| \overset{(3.2)}{=} |\bigcup_{i=1}^{k}(M \cap N(C_i))| = \sum_{i=1}^{k} |B_i| \overset{(3.5)}{\leq} \sum_{i=1}^{k}(|A_i| + \delta(G)) \tag{3.6}$$
$$= \sum_{i=1}^{k} |A_i| + \sum_{i=1}^{k} \delta(G) \leq |C| + \tau(G)\delta(G) = \tau(G)[1 + \delta(G)]$$

□

When $\delta(G) > 0$ it is not difficult to characterize the graphs G such that $\alpha(G) = \tau(G)[1 + \delta(G)]$. Let τ and δ be positive numbers and let $H^{-}(\tau, \delta)$ and $H^{+}(\tau, \delta)$ be the graphs



on the vertex set $V(H^{-}(\tau, \delta)) = V(H^{+}(\tau, \delta)) = V_0 \cup V_1 \cup \cdots \cup V_\tau$ where $V_0 = \{v_1, v_2, \ldots, v_\tau\}$, $V_i = \{v_1^i, \ldots, v_{\delta+1}^i\}$ for all $i = 1, \ldots, \tau$ and edge sets

$$E(H^{-}(\tau, \delta)) = (\bigcup_{i=1}^{\tau}\{\{v_i, v_j^i\} \mid 1 \leq j \leq \delta + 1\})$$

and

$$E(H^{+}(\tau, \delta)) = E(H^{-}(\tau, \delta)) \cup \{\{v_i, v_j\} \mid 1 \leq i \neq j \leq \tau\}.$$

A set of edges in a graph G is called independent or a *matching* if no two of them have a vertex in common. A pairing by an independent set of edges of all the vertices of a graph G is called a *perfect matching*.

COROLLARY (3.7). *Let G be a graph without isolated vertices.*

(*i*) *If $\delta(G) > 0$, then $\alpha(G) = \tau(G)[1 + \delta(G)]$ if and only if*

$$E(H^{-}(\tau, \delta)) \subseteq E(G) \subseteq E(H^{+}(\tau, \delta)),$$

where $\tau = \tau(G)$ and $\delta = \delta(G)$.

(*ii*) *If $\delta(G) = 0$ and $\alpha(G) = \tau(G)$, then G has a perfect matching.*

Proof. We use the same notation as in the proof of Theorem (3.1).

(i) Since $\delta(G) > 0$ and $\alpha(G) = \tau(G)[1 + \delta(G)]$, then using equation (3.6) we can conclude that $k = \tau(G)$.

Following the proof of Theorem (3.1), we have that $|C \cap M'| \leq 1$ for all M' maximal stable sets. Moreover, for all $u \in C$ there exists a M' maximal stable set with $C \cap M' = \{u\}$. Thus, the equation (3.3) reduces to $|M \cap N(u)| \leq 1 + \delta(G)$ for all $u \in C$, where $M = V(G) \setminus C$. On the other hand, since $M = \cup_{u \in C}(M \cap$

$N(u))$ and $\alpha(G) = \tau(G)[1 + \delta(G)]$, then $|M \cap N(u)| = 1 + \delta(G)$ for all $u \in C$ and $(M \cap N(v)) \cap (M \cap N(u)) = \emptyset$ for all $u \neq v \in C$. Furthermore, since M is a stable set, then $E(H^-(\tau(G), \delta(G))) \subseteq E(G) \subseteq E(H^+(\tau(G), \delta(G)))$.

Finally, note that if $E(H^-(\tau, \delta)) \subseteq E(G) \subseteq E(H^+(\tau, \delta))$ for some $\tau > 0$ and $\delta > 0$, then clearly $\alpha(G) = \tau(G)[1 + \delta(G)]$.

(ii) Following the proof of Theorem (3.1) we have that *(ii)* reduces to prove that for all $i = 1, \ldots, k$ the induced subgraph $G_i = G[A_i \cup B_i]$ has a perfect matching, that is, $\nu(G_i) = |A_i| = |B_i|$ for all $i = 1, \ldots, k$.

Since G_i is a bipartite graph (A_i and B_i are stable sets of G), then by Konig's theorem $\nu(G_i) = \tau(G_i)$. Hence, we only need to prove that $|A_i| = |B_i|$ and $\tau(G_i) = |A_i|$ for all $i = 1, \ldots, k$.

First, since $C = \sqcup_{i=1}^{k} A_i$ and $M = \sqcup_{i=1}^{k} B_i$, then $\sum_{i=1}^{k} |A_i| = \tau(G) = \alpha(G) = \sum_{i=1}^{k} |B_i|$. On the other hand, since $\delta(G) = 0$, then the equation (3.5) reduces to $|A_i| \leq |B_i|$ and therefore $|A_i| = |B_i|$ for all $i = 1, \ldots, k$.

Finally, we will prove that $\tau(G_i) = |A_i|$ for all $i = 1, \ldots, k$. Since A_i is a vertex cover of G_i, then $\tau(G_i) \leq |A_i|$. Furthermore, if $\tau(G_i) < |A_i|$, then there exist a stable set N of G_i with $|N| > |A_i|$. Since $M \cap N(\cup_{j=1}^{i} C_j) = \cup_{j=1}^{i} (M \cap N(C_i)) = \cup_{j=1}^{i} M \setminus M_j$, then $N \cup (T_i \setminus A_i) = (N \cap A_i) \cup M_i \cup (C_i \setminus A_i) \cup (N \cap B_i) \subset T_i \cup (N \cap B_i)$ is a stable set of $(N(C_i \setminus A_i) \cap B_i = \emptyset)$ of G with $|T_i| - |A_i| + |N| > |T_i| = \alpha(G)$ vertices; a contradiction. □

(3.1) B-graphs. A graph is called a *B-graph* if every vertex belongs to a maximum stable set (that is, to a stable set of largest size). This concept was introduced by Berge in [1].

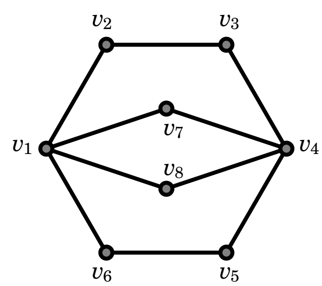
The *σ_v-cover* number of a graph, denoted by $\sigma_v(G)$, is the maximum natural number m, such that every vertex of G belongs to a maximal independent set with at least m vertices. Clearly, G is a B-graph if and only if $\alpha(G) = \sigma_v(G)$ if and only if $\delta(G) = 0$, where $\delta(G) = \alpha(G) - \sigma_v(G)$.

Now we define two invariants that measure when a graph is a B-graph or a τ-critical graph. Let

$$\alpha_{\text{core}}(G) = \bigcap_{|\mathcal{M}_i| = \alpha(G)}^{\textit{stable set}} \mathcal{M}_i \quad \text{and} \quad \tau_{\text{core}}(G) = \bigcap_{|\mathcal{C}_i| = \tau(G)}^{\textit{vertex cover}} \mathcal{C}_i,$$

be the intersection of all the maximum stable sets and of all the minimum vertex covers of G, respectively. Also, let $B_{\alpha\cap\tau} = V(G) \setminus (\alpha_{\text{core}}(G) \cup \tau_{\text{core}}(G))$.

Example (3.1.1). To illustrate the concepts of $\alpha_{\text{core}}(G)$, $\tau_{\text{core}}(G)$ and $B_{\alpha\cap\tau}$ consider the following graph:



since $\alpha(G) = 4$, $\tau(G) = 4$ and $\{v_2, v_5, v_7, v_8\}$, $\{v_2, v_6, v_7, v_8\}$, $\{v_3, v_5, v_7, v_8\}$, $\{v_3, v_6, v_7, v_8\}$ are the maximum stable sets of G, then

- $\alpha_{\text{core}}(G) = \{v_7, v_8\}$,
- $\tau_{\text{core}}(G) = \{v_1, v_4\}$, and
- $B_{\alpha\cap\tau} = \{v_2, v_3, v_5, v_6\}$.

Since M is a maximum stable set of G if and only if $C = V(G) \setminus M$ is a minimum vertex cover of G, then G is a B-graph if and only if $V(G) = \bigcup_{|\mathcal{M}_i|=\alpha(G)}^{\textit{stable set}} \mathcal{M}_i$ if and only if $\tau_{\text{core}}(G) = \bigcap_{|\mathcal{C}_i|=\tau(G)}^{\textit{vertex cover}} \mathcal{C}_i = \emptyset$. Similarly, since a graph is *τ-critical* if and only if $\tau(G - v) < \tau(G)$ for all $v \in V(G)$ if and only if there exists a maximum stable set M_v of G such that $v \notin M_v$ for all $v \in v(G)$, then G is a τ-critical graph if and only if $\alpha_{\text{core}}(G) = \emptyset$.

PROPOSITION (3.1.2). *Let G be a graph, then*

$$V(G) = \alpha_{\text{core}}(G) \sqcup \tau_{\text{core}}(G) \sqcup B_{\alpha\cap\tau},$$

furthermore

(*i*) $G[\alpha_{\text{core}}(G)]$ *is a trivial graph,*
(*ii*) $N(\alpha_{\text{core}}(G)) \subseteq \tau_{\text{core}}(G)$*,*
(*iii*) $G[B_{\alpha\cap\tau}]$ *is both a τ-critical graph as well as a B-graph without isolated vertices, and*
(*iv*) $\alpha(G) - |\alpha_{\text{core}}(G)| \leq \tau(G) - |\tau_{\text{core}}(G)|$.

Proof. Firstly, is clear that $\alpha_{\text{core}}(G) \cap \tau_{\text{core}}(G) = \emptyset$. Also, by the definition of $B_{\alpha\cap\tau}$ it is clear that $\alpha_{\text{core}}(G) \cap B_{\alpha\cap\tau} = \emptyset$ and $\tau_{\text{core}}(G) \cap B_{\alpha\cap\tau} = \emptyset$.

(i) Since $\alpha_{\text{core}}(G)$ is the intersection of stable sets, then $\alpha_{\text{core}}(G)$ is a stable set and therefore $G[\alpha_{\text{core}}(G)]$ is a trivial graph.

(ii) Since $\tau_{\text{core}}(G) = V(G) \setminus \bigcup_{|\mathcal{M}_i|=\alpha(G)}^{\textit{stable set}} \mathcal{M}_i$ and $\alpha_{\text{core}}(G) \subset V(G) \setminus \tau_{\text{core}}(G)$, then $\alpha_{\text{core}}(G)$ is the set of isolated vertices of $G[V(G) \setminus \tau_{\text{core}}(G)]$. Therefore $N(\alpha_{\text{core}}(G)) \subseteq \tau_{\text{core}}(G)$.

(iii) Since $\alpha(G[B_{\alpha\cap\tau}]) = \alpha(G) - |\alpha_{\text{core}}(G)|$, $\tau(G[B_{\alpha\cap\tau}]) = \tau(G) - |\tau_{\text{core}}(G)|$ and $B_{\alpha\cap\tau} = V(G) \setminus (\alpha_{\text{core}}(G) \cup \tau_{\text{core}}(G))$, then $\alpha_{\text{core}}(B_{\alpha\cap\tau}) = \emptyset$ and $\tau_{\text{core}}(B_{\alpha\cap\tau}) = \emptyset$. Therefore $G[B_{\alpha\cap\tau}]$ is a τ-critical graph and a B-graph without isolated vertices.

(iv) Since $G[B_{\alpha\cap\tau}]$ is a B-graph, then $\delta(G[B_{\alpha\cap\tau}]) = 0$. Therefore, applying Theorem (3.1) to $G[B_{\alpha\cap\tau}]$,

$$\alpha(G) - |\alpha_{\text{core}}(G)| = \alpha(G[B_{\alpha\cap\tau}]) \leq \tau(G[B_{\alpha\cap\tau}]) = \tau(G) - |\tau_{\text{core}}(G)|. \qquad \square$$

Remark (3.1.3). If v is an isolated vertex, then $v \in \alpha_{\text{core}}(G)$, and if $\deg(v) > \tau(G)$, then v does not belong to any stable set with $\alpha(G)$ vertices and therefore $v \in \tau_{\text{core}}(G)$. Note that in general the induced graph $G[B_{\alpha\cap\tau}]$ is not necessarily connected.

COROLLARY (3.1.4). ([1], Proposition 7) *If G is a B-graph without isolated vertices, then G is a τ-critical graph.*

Proof. Since G is a B-graph, then $\tau_{\text{core}}(G) = \emptyset$. Thus, by Proposition (3.1.2) (ii), $N(\alpha_{\text{core}}(G)) = \tau_{\text{core}}(G) = \emptyset$. Moreover, since G has no isolated vertices, then $\alpha_{\text{core}}(G) = \emptyset$. Therefore G is a τ-critical graph. $\square$

Remark (3.1.5). The bound of Proposition (3.1.2) (iv) improves the bound given in [9], Theorem 2.11, for the number of vertices in $\alpha_{\text{core}}(G)$.

Their result states that if G is a graph of order n and

$$\alpha(G) > \frac{n + k - \min\{1, |N(\alpha_{\text{core}}(G))|\}}{2}, \text{ for some } k \geq 1,$$

then $|\alpha_{\text{core}}(G)| \geq k + 1$. Moreover, if $n + k - \min\{1, |N(\alpha_{\text{core}}(G))|\}$ is even, then $|\alpha_{\text{core}}(G)| \geq k + 2$. Our result states that if G is a graph of order n and $\alpha(G) \geq \frac{n+k'}{2}$ for some $k' \geq 0$, then

$$|\alpha_{\text{core}}(G)| \overset{\text{(iv)}}{\geq} \alpha(G) - \tau(G) + |\tau_{\text{core}}(G)| = 2\alpha(G) - n + |\tau_{\text{core}}(G)| \geq k' + |\tau_{\text{core}}(G)|.$$

In order to compare both bounds we can write their bound in the following equivalent way: If G is a graph of order n, $|N(\alpha_{\text{core}}(G))| = 0$ ($|N(\alpha_{\text{core}}(G))| \geq 1$) and

$$\alpha(G) \geq \begin{cases} \frac{n+(k+1)}{2} \; (\frac{n+(k+1)}{2}) \text{ if } n+k \text{ is odd,} \\ \frac{n+(k+2)}{2} \; (\frac{n+k}{2}) \text{ if } n+k \text{ is even.} \end{cases}$$

for some $k \geq 1$, then

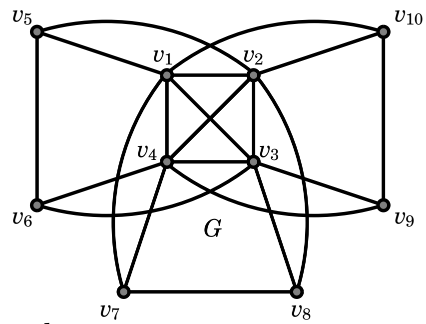
$$|\alpha_{\text{core}}(G)| \geq \begin{cases} k+1 \, (k+2) \text{ if } n+k \text{ is odd,} \\ k+2 \, (k+1) \text{ if } n+k \text{ is even.} \end{cases}$$

Since $|N(\alpha_{\text{core}}(G))| \leq |\tau_{\text{core}}(G)|$ (Proposition (3.1.2) (ii)), then our bound improves their bound. Furthermore, the bounds are equivalent if and only if $|N(\alpha_{\text{core}}(G))| = |\tau_{\text{core}}(G)| = 0, 1$.

3.1.6. *Conjectures.* In this section we present a conjecture that generalizes the result obtained from Theorem 3.1 when G is a B-graph. Before stating the conjecture we will introduce a new graph invariant.

Definition (3.1.7). The ω_e*-clique covering* number of G, denoted by $\omega_e(G)$, is the greatest natural number m so that every edge in G belongs to a clique of size at least m.

Example (3.1.8). In order to illustrate the previous concept consider the following graph:



For this graph we have that:

- $\omega(G) = \alpha(\overline{G}) = 4$ because $\{v_1, v_2, v_3, v_4\}$ is a clique of G,
- $\omega_e(G) = 2$ because the edge $\{v_5, v_6\}$ is not in any induced $\mathcal{K}_3$ of G,

• $\alpha(G) = \omega_e(\overline{G}) = 4$ because $\{v_1, v_6, v_8, v_9\}$, $\{v_2, v_6, v_7, v_9\}$, $\{v_3, v_5, v_7, v_{10}\}$, and $\{v_4, v_5, v_8, v_{10}\}$ are stable sets of G, and

• $\sigma_v(\overline{G})=3$ because $\{v_{i-1}, v_i, v_{10-2i}\}$ for $i=1, 2, 3$ and $\{v_1, v_2, v_5\}$, $\{v_1, v_4, v_7\}$, $\{v_3, v_4, v_9\}$ are cliques of G.

The ω_e-*clique covering* and the σ_v-*cover* numbers of G satisfy the following two identities:

$$\omega_e(G) \leq \sigma_v(\overline{G}) \leq \omega(G) \text{ and } \omega_e(\overline{G}) \leq \sigma_v(G) \leq \alpha(G).$$

Conjecture (3.1.9). Let G be a B-graph ($\sigma_v(G) = \alpha(G)$) without isolated vertices, then

$$\omega_e(G)\sigma_v(G) \leq |V(G)|.$$

Furthermore, for all maximum stable sets M, there exist disjoint sets

$$A_j \subset V(G) \text{ for all } j = 1, \ldots, |M|,$$

such that

(i) $|M \cap A_j| = 1$ for all $j = 1, \ldots, |M|$,

(ii) $G[A_j]$ is a clique of order $\omega_e(G)$ for all $i = 1, \ldots, |M|$.

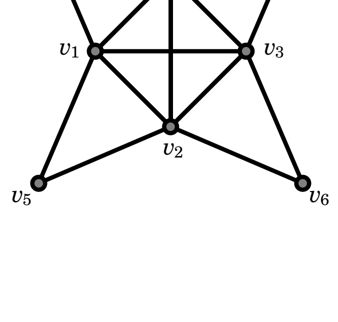
Remark (3.1.10). If G is a graph without isolated vertices, then $\omega_e(G) \geq 2$. Thus, if Conjecture (3.1.9) holds, then $2\alpha(G) \leq |V(G)|$ when G is a B-graph without isolated vertices. On the other hand, Theorem (3.1) implies that if G is a B-graph without isolated vertices, then $\alpha(G) \leq \tau(G)$. Since $\tau(G) = |V(G)| - \alpha(G)$, then Conjecture (3.1.9) implies the bound given in Theorem (3.1) when G is a B-graph.

Remark (3.1.11). A weaker version of Conjecture (3.1.9) is given in [7] and [8]. In these papers the authors prove that, if G is a graph with n vertices such that every vertex belongs to a clique of cardinality $q + 1$ and a stable set of cardinality $p + 1$, then $|V(G)| \geq p + q + \sqrt{4pq}$. Using our terminology, this bound can be writen as

$$\sigma_v(G) + \sigma_v(\overline{G}) - 2 + 2\sqrt{(\sigma_v(G) - 1)(\sigma_v(\overline{G}) - 1)} \leq |V(G)|.$$

Since $\sigma_v(G)+\sigma_v(\overline{G})-2+2\sqrt{(\sigma_v(G) - 1)(\sigma_v(\overline{G}) - 1)} \leq \max\{4(\sigma_v(G)-1), 4(\sigma_v(\overline{G})-1)\}$, then in comparison with our bound, this bound is a bad lower bound for the number of vertices of a B-graph G.

Remark (3.1.12). If we do not assume that G is a B-graph, then Conjecture (3.1.9) is false. To see this fact, consider the following graph:



For this graph we have,

- $\alpha(G) = 4$ because $\{v_5, v_6, v_7, v_8\}$ is a stable set,
- $\sigma_v(G) = 3$ because $\{v_1, v_6, v_7\}$, $\{v_2, v_7, v_8\}$, $\{v_3, v_5, v_8\}$, and $\{v_4, v_5, v_6\}$ are stable sets,
- $\omega_e(G) = 3$ because all the edges are in an induced $\mathcal{K}_3$,
- $\omega(G) = 4$, $\sigma_e(G) = 3$ and $\omega_v(G) = 3$ because $\overline{G} \cong G$, and
- G is not a B-graph because $\sigma_v(G) = 3 \neq 4 = \alpha(G)$.

However,

$$\omega_e(G)\sigma_v(G) = (3)(3) = 9 > 8 = n.$$

Hypergraphs. A *hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (V, \mathcal{E})$ where V is a set of elements, called vertices, and $\mathcal{E}$ is a set of non-empty subsets of V called hyperedges. We say that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a r-*uniform* hypergraph if $|E| = r$ for all $E \in \mathcal{E}$. A vertex $v \in V$ of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called isolated if $v \notin E$ for all $E \in \mathcal{E}$.

A subset M of vertices of $\mathcal{H}$ is called a *stable* set if no two vertices in M belong to a hyperedge of $\mathcal{H}$. We say that M is a *maximal stable* set if it is maximal with respect to inclusion. The *stability number* of a hypergraph $\mathcal{H}$ is given by

$$\alpha(\mathcal{H}) = \max\{|M| \mid M \subset V(\mathcal{H}) \text{ is a stable set in } \mathcal{H}\}.$$

The σ_v-*cover* number of a hypergraph $\mathcal{H}$, denoted by $\sigma_v(\mathcal{H})$, is the maximum natural number m such that every vertex of $\mathcal{H}$ belongs to a maximal independent set of $\mathcal{H}$ with at least m vertices.

The next conjecture was stated in [13], Conjecture 3.2.12.

Conjecture (3.1.13). Let $\mathcal{H} = (V, \mathcal{E})$ be a r-uniform hypergraph without isolated vertices. If $\sigma_v(\mathcal{H}) = \alpha(\mathcal{H})$, then

$$r\sigma_v(\mathcal{H}) \leq |V|.$$

The last conjecture follows from Conjecture (3.1.9) by the following argument: Let $\mathcal{H}$ be a hypergraph and consider the graph $G(\mathcal{H})$ defined on the same vertex set of $\mathcal{H}$ and for which $v_1, v_2 \in G(\mathcal{H})$ are adjacent if and only if they are adjacent in $\mathcal{H}$.

Clearly $G(\mathcal{H})$ has the same stability number as $\mathcal{H}$. Also, observe that $G(\mathcal{H})$ and $\mathcal{H}$ have the same σ_v-cover number. Moreover, if $\mathcal{H}$ is a r-uniform hypergraph, then $r \leq \omega_e(G(\mathcal{H}))$. Applying Conjecture (3.1.9) to the graph $G(\mathcal{H})$, we have that

$$r\sigma_v(\mathcal{H}) \leq \omega_e(G(\mathcal{H}))\sigma_v(G(\mathcal{H})) \leq |V|.$$

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SOME REMARKS ON ROBIN'S INEQUALITY

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ABSTRACT. In this paper we give an elementary and short proof that the Robin inequality (R): $\sigma(n) < e^{\gamma} n \log\log n$ is true under some condition. Namely, we prove that for every positive integers $n = 2^{\alpha} m$, $\alpha \geq 2$, $(2, m) = 1$ and $m = m_1 M > \frac{1}{2} e^{e^9}$ the inequality (R) is true if the integer m_1 of the form $m_1 = p_1^{\alpha_1} p_2^{\alpha_2}$, satisfies the inequality $I(m_1) = \prod_{j=1}^{2} (1 - 1/p_j^{1+\alpha_j}) < \frac{49}{50}$. For $n = 2m$, $(2, m) = 1$ where $m > \frac{3^9}{2}$ and for such m inequality (R) has been proved in our paper [4]. The Robin inequality (R) for all positive integers $n \geq 5041$ implies Riemann Hypothesis. The positive integers $n \in [5041, e^{e^9})$ also satisfy the inequality (R). This fact has been checked by computer calculation.

1. Introduction

The Riemann zeta function $\zeta(s)$ for $s = \sigma + it$ is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges for $\sigma > 1$ and it has analytic continuation to the complex plane with one singularity, a simple pole with residue 1. In 1859 Riemann [11] conjectured that the nonreal zeros of the Riemann zeta function $\zeta(s)$ all lie on the line $s = \frac{1}{2} + it$.

The connection of the Riemann hypothesis with prime numbers has been considered by Gauss. Let

$$\Pi(x) = \sum_{1<p\leq x} 1,$$

then it is well-known that the Riemann hypothesis is equivalent to the assertion that for each $\varepsilon > 0$ there is a positive constant $C = c(\varepsilon)$ such that

$$|\Pi(x) - Li(x)| \leq c(\varepsilon) x^{\frac{1}{2}+\varepsilon},$$

where

$$Li(x) = \int_2^x \frac{dt}{\log t}.$$

The Riemann zeta-function is a special case of an L-function. These L-functions are connected with many important and difficult problems in number theory, algebraic geometry, topology, representation theory and modern physics, see: Berry and Keating [1], Katz and Sarnak [7], Murty [9]. It is known that the Riemann hypothesis is related to estimates of error terms

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associated with the Farey sequence of reduced fractions in the unit interval. Important examples connected with these problem have been given by Yoshimoto in the papers [14], [15], [16] and Kanemitsu and Yoshimoto [5], [6].

In 1984 Robin [12] proved a very interesting and important criterion:

CRITERION (1.1) (Robin). *The Riemann Hypothesis is true if and only if*

$$\sigma(n) < e^{\gamma} n \log\log n, \tag{R}$$

for all positive integers $n \geq 5041$, *where*

$$\sigma(n) = \sum_{d|n} d,$$

and $\gamma \approx 0.57728$ *is Euler's constant.*

In 2002 Lagarias [8] proved the following criterion:

CRITERION (1.2) (Lagarias). *Let* $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. *The Riemann Hypothesis is true if and only if*

$$\sigma(n) \leq H_n + \exp H_n \log H_n, \tag{L}$$

for each positive integer $n \geq 1$, *and with equality in* (L) *only for* $n = 1$.

In the same paper Lagarias proved that for all positive integers $n \geq 3$ we have

$$e^{\gamma} n \log\log n \leq \exp H_n \log H_n. \tag{L$_1$}$$

From (L), (L_1) and (R) follows that Lagarias' criterion is an extension of the Robin criterion. Many others criterions and important results connected with the Riemann hypothesis have been proved and these results have been described by Conrey in [2].

In our paper [4] we gave an elementary proof that the Robin inequality (R) is true for all even positive integers $n = 2m$, $(2, m) = 1$ such that $m > \frac{3^9}{2}$ is odd. Namely, the following result has been proved in [4]:

THEOREM (1.3) ([4]). *Let* $n = 2m$, $(2, m) = 1$. *Then for all odd positive integers* $m > \frac{3^9}{2}$ *we have*

$$\sigma(2m) < \frac{39}{40} e^{\gamma} 2m \log\log 2m, \tag{1.4}$$

and

$$\sigma(m) < e^{\gamma} m \log\log m. \tag{1.5}$$

In the proof of this result use was made of the following Rosser-Schoenfeld's inequality ([13], [10], p. 169):

$$\frac{n}{\varphi(n)} \leq e^{\gamma} \left(\log\log n + \frac{2.5}{e^{\gamma} \log\log n} \right), \tag{1.6}$$

where φ is the Euler totient function and Theorem (1.3) is true for all positive integers $n \geq 3$ except $n = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 19 \times 23$. In this case the constant $c = 2.5$ must be replaced by the constant $c_1 = 2.50637 < 2.51$. Hence, for all positive integers $n \geq 3$ we have

$$\frac{n}{\varphi(n)} < e^{\gamma} \log\log n \left(1 + \frac{2.51}{e^{\gamma}(\log\log n)^2}\right). \tag{1.7}$$

Let $n = \prod_{j=1}^{k} p_j^{\alpha_j}$, $p_j \in P$, where P is the set of prime numbers and $\alpha_j \geq 1$ are integers for each j =1, 2, ..., k.

The following identity has been used in the proof of the Theorem (1.3):

$$\frac{\sigma(n)}{n} = \prod_{j=1}^{k}\left(1 - \frac{1}{p_j^{1+\alpha_j}}\right)\frac{n}{\varphi(n)}, \tag{1.8}$$

where σ is the sum divisors function and φ is the Euler totient function.

We note that in a recently published paper [3] by Choie, Lichardopol, Moree and Solé has been proved that if $n \geq 37$ does not satisfy Robin's criterion it must be even and is neither squarefree nor squarefull, moreover that n must be divisible by a fifth power > 1. As a consequence they proved that the Riemann Hypothesis holds true iff every natural number divisible by a fifth power > 1 satisfies Robin's inequality.

In this paper under some condition we give a short proof of the Robin inequality (R) for the remains case. Namely, we prove the following theorem:

THEOREM (1.9). *The Robin inequality* (R) *is true for all positive integers* $n = 2^{\alpha}m$, *where* $(2, m) = 1$, $\alpha \geq 2$ *and* $m = m_1M > \frac{1}{2}e^{e^9}$, *if for odd positive integer* m_1, *such that* $\omega(m_1) = 2$, *where* $\omega(m_1)$ *is the number of all distinct primes of* m_1, *the inequality*

$$I(m_1) = \prod_{j=1}^{2}\left(1 - \frac{1}{p_j^{1+\alpha_j}}\right) < \frac{49}{50}$$

is satisfied.

2. Basic lemmas

In the proof of the Theorem (1.9) we use of the following Lemmas:

LEMMA (2.1). *Let* $n = 2^{\alpha}m$, $(2, m) = 1$, *and* $\omega(m)$ *is the number of distinct primes of* m. *If* $\omega(m) = 1$, *then for every odd positive integer* $m > \frac{1}{4}e^{e^2}$ *and each fixed integer* $\alpha \geq 2$, *we have*

$$\sigma(2^{\alpha}m) < e^{\gamma}2^{\alpha}m \log\log 2^{\alpha}m. \tag{2.2}$$

LEMMA (2.3) ([4], Thm. 2). *If for each odd positive integer* $m > m_0$ *the following inequality*

$$\sigma\,(2m) < \frac{3}{4}e^{\gamma}2m \log\log 2m \tag{2.4}$$

is true, then for all integers $n = 2^{\alpha}m$, $(2, m) = 1$, $m > m_0$ *and every fixed integer* $\alpha \geq 2$ *we have*

$$\sigma(2^{\alpha}m) < e^{\gamma}2^{\alpha}m \log\log 2^{\alpha}m. \tag{2.5}$$

Proof of Lemma (2.1). First we note that if $n = 2^{\alpha}m$, $(2, m) = 1$ and $\omega(m) = 1$, $m > \frac{1}{4}e^{e^2}$ then we have

$$m = p_1^{\alpha_1}, \alpha \geq 2 \quad \text{and} \quad p_1^{\alpha_1} > \frac{1}{4}e^{e^2}. \tag{2.6}$$

Since

$$\sigma(2^\alpha m) = \sigma(2^\alpha)\sigma(m) = (2^{\alpha+1}-1)\sigma(p_1^{\alpha_1}) = \left(2^{\alpha+1}-1\right)\frac{p_1^{1+\alpha_1}-1}{p_1-1}, \tag{2.7}$$

then by (2.6), (2.7) and the fact that $p_1 - 1 \geq \frac{2}{3}p_1$ it follows that

$$\sigma\left(2^\alpha m\right) < 2^{\alpha+1} \times \frac{3}{2}p_1^{\alpha_1} = 3 \times 2^\alpha m. \tag{2.8}$$

On the other hand by the assumption it follows that $2^\alpha m > 2^2 \times \frac{1}{4}e^{e^2} > e^{e^2}$. Hence, we have

$$e^\gamma \log\log 2^\alpha m > 1.6 \log\log e^{e^2} > 1.6 \times 2 = 3.2 > 3. \tag{2.9}$$

From (2.9) and (2.8) we get

$$\sigma\left(2^\alpha m\right) < e^\gamma 2^\alpha m \log\log 2^\alpha m,$$

and the proof of the Lemma (2.1) is finished. □

LEMMA (2.10). *Let m be an odd positive integer and let $m = \prod_{j=1}^{k} p_j^{\alpha_j}$, If $I(m) + \frac{2.51}{e^\gamma(\log\log 2m)^2} < 1$, for $m > m_o$, where $I(m) = \prod_{j=1}^{k}\left(1 - \frac{1}{p_j^{1+\alpha_j}}\right)$ then*

$$\frac{\sigma(2m)}{2m} < \frac{3}{4}e^\gamma \log\log 2m. \tag{2.11}$$

Proof. From the identity (1.8), Rosser-Schoenfeld's inequality (1.6) and the assumption of the Lemma (2.10) it follows that

$$\frac{\sigma(2m)}{2m} < \frac{3}{4}I(m)\left(1 + \frac{2.51}{e^\gamma(\log\log 2m)^2}\right)e^\gamma \log\log 2m. \tag{2.12}$$

Since $I(m) < 1$ then by the assumption of the Lemma (2.10) it follows that

$$I(m)\left(1 + \frac{2.51}{e^\gamma(\log\log 2m)^2}\right) < I(m) + \frac{2.51}{e^\gamma(\log\log 2m)^2} < 1. \tag{2.13}$$

Hence, from (2.13) and (2.12) we obtain the inequality 2.11. □

LEMMA (2.14). *Let $n = 2m_1$ and m_1 be an odd positive integer such that $\omega(m_1) = 2$. Then for $m_1 > \frac{1}{2}e^{e^3}$ we have*

$$\frac{\sigma\left(2m_1\right)}{2m_1} < \frac{3}{4}e^\gamma \log\log 2m_1. \tag{2.15}$$

Proof. From the assumption of Lemma (2.14) we have that $m_1 = p_1^{\alpha_1}p_2^{\alpha_2}$ and we get

$$\frac{\sigma(2m_1)}{2m_1} = \frac{\sigma(2)\sigma(m_1)}{2m_1} = \frac{3}{2}\frac{(p_1^{1+\alpha_1}-1)(p_2^{1+\alpha_2}-1)}{p_1^{\alpha_1}p_2^{\alpha_2}(p_1-1)(p_2-1)}. \tag{2.16}$$

Since

$$p_1 - 1 \geq \frac{2}{3}p_1, \quad p_2 - 1 \geq \frac{4}{5}p_2, \quad p_1^{1+\alpha_1} - 1 < p_1^{1+\alpha_1}, \quad p_2^{1+\alpha_2} - 1 < p_2^{1+\alpha_2} \tag{2.17}$$

then by (2.17) and (2.16) it follows that

$$\frac{\sigma(2m_1)}{2m_1} < \frac{3}{2} \times \frac{3}{2} \times \frac{5}{4} = \frac{3}{4} \times \frac{15}{4}. \tag{2.18}$$

On the other hand since $e^{\gamma} > 1.6$ and $2m_1 > e^{e^3}$, hence

$$e^{\gamma} \log\log 2m_1 > 1.6 \times 3 = 4.8 > \frac{15}{4}. \tag{2.19}$$

From (2.18) and (2.19) we get

$$\frac{\sigma(2m_1)}{2m_1} < \frac{3}{4} \times \frac{15}{4} < \frac{3}{4} e^{\gamma} \log\log 2m_1, \tag{2.20}$$

and the proof of Lemma (2.14) is complete. □

3. Proof of the Theorem (1.9)

From Lemma (2.1), Lemma (2.3) and Lemma (2.14) follows that we can assume that $\omega(m) = k > 2$. Applying Lemma (2.3) we prove the inequality (2.4) for all odd positive integers $m > \frac{1}{2} e^{e^9}$.

Let $n = 2m$, $(2, m) = 1$ and let

$$m = \prod_{j=1}^{k} p_j^{\alpha_j}, \quad I(m) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j^{1+\alpha_j}}\right), \quad k = \omega(m) > 2. \tag{3.1}$$

From the identity (1.8) we obtain

$$\frac{\sigma(2m)}{2m} = \left(1 - \frac{1}{2^2}\right) I(m) \frac{2m}{\varphi(2m)} = \frac{3}{4} I(m) \frac{2m}{\varphi(2m)}.$$

Now, applying to $\frac{2m}{\varphi(2m)}$ the Rosser-Schoenfeld inequality (1.7) we get

$$\frac{\sigma(2m)}{2m} < \frac{3}{4} \left(I(m) + \frac{2.51}{e^{\gamma}(\log\log 2m)^2} \right) e^{\gamma} \log\log 2m. \tag{3.2}$$

On the other hand by the assumption that $m > \frac{1}{2} e^{e^9}$ and the inequality $e^{\gamma} > 1.6$, $\alpha \geq 2$, it follows that

$$e^{\gamma} \left(\log\log 2m\right)^2 > 1.6 \left(\log\log 2 \times \frac{1}{2} e^{e^9}\right)^2 > 1.6 \times 81 > 50 \times 2.51. \tag{3.3}$$

From (3.3) we get

$$\frac{2.51}{e^{\gamma} \left(\log\log 2m\right)^2} < \frac{1}{50}. \tag{3.4}$$

Since $\omega(m) > 2$, then we have $m = m_1 M$, where $\omega(m_1) = 2$. Moreover, we have

$$I(m) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j^{1+\alpha_j}}\right) = I(m_1) I(M). \tag{3.5}$$

By the assumption of the Theorem (1.9) it follows that $I(m_1) < \frac{49}{50}$, thus from (3.5) we obtain

$$I(m) < \frac{49}{50} I(M) < \frac{49}{50}, \tag{3.6}$$

because $I(M) < 1$.

From (3.6) and (3.4) we have

$$I(m) + \frac{2.51}{e^{\gamma}(\log\log 2m)^2} < \frac{49}{50} + \frac{1}{50} = 1. \tag{3.7}$$

By (3.7), (3.2) and Lemma (2.10) it follows that

$$(3.8)\quad \frac{\sigma(2m)}{2m} < \frac{3}{4}e^{\gamma}\log\log 2m \times \left(I(m) + \frac{2.51}{e^{\gamma}(\log\log 2m)^2}\right) < \frac{3}{4}e^{\gamma}\log\log 2m.$$

From (3.8) and Lemma (2.3) we obtain that the proof of the Theorem (1.9) is complete. □

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ON THE CHARACTERIZATION OF THE KNEADING SEQUENCES ASSOCIATED TO INJECTIVE LORENZ MAPS OF THE INTERVAL AND TO ORIENTATION PRESERVING HOMEOMORPHISMS OF THE CIRCLE

Dedicated to Michael Robert Hermann (1942-2000)

RAFAEL LABARCA AND LAUTARO VÁSQUEZ

ABSTRACT. In this paper we characterize the kneading sequences associated to injective Lorenz maps of the interval and to orientation preserving homeomorphisms of the circle.

1. Introduction

It is well known that the evolution of many processes can be dynamically explained by the iteration of a map on a interval (see for instance [7], [19]). In several other situations the dynamics of a higher dimensional system can be reduced to the study of a map defined on an interval (see for instance [2], [6], [9], [15], [14]). This is the case for the geometric model which R.F. Williams [20] proposes to explain the existence of the strange attractor numerically detected by E. N. Lorenz [17] for a quadratic vector field defined on $\mathbb{R}^3$. By assuming that certain foliations remain invariant by the Poincaré map associated to a doubly homoclinic loop of the vector field, the behavior of the flow in a neighborhood of the loop can be understood from the iteration of a map defined on an interval. In figure 1 we give a sketch of the geometric Lorenz attractor and in figure 2 we represent the one dimensional model associated to the attractor according to the orientation of the vector field along the homoclinic loop.

The increasing and discontinuous one dimensional map given in figure 2 was used by Guckenheimer and Williams ([6]) to show the existence of uncountable many classes of non-equivalent geometric Lorenz attractors. The evidence of the non-equivalence follows from the kneading sequences associated to these one dimensional maps: Each one of these maps is semi-conjugated to an ordered subshift of the shift map, σ, defined on the set, Σ_2, of sequences $\theta\colon \mathbb{N}_0 \to \{0, 1\}$ endowed with the metric $d(\theta, \xi) = \frac{1}{2^n}$, where $n = \min\{k \in \mathbb{N} : \theta_k \neq \xi_k\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In fact, if we consider in Σ_2 the lexicographical order ($\theta < \alpha$ if $\theta_i = \alpha_i$, $i = 0, 1, \ldots, k-1$ and $\theta_k = 0$, $\alpha_k = 1$) we can define the minimal (resp. maximal) sequences in Σ_2 as those $a \in \Sigma_0$ (resp. $b \in \Sigma_1$) such that $\sigma^i(a) \geq a$ (resp. $\sigma^i(b) \leq b$) for all $i \in \mathbb{N}$ (here $\Sigma_i = \{\theta \in \Sigma_2 : \theta(0) = i\}$, $i = 0, 1$). For a, $b \in \Sigma_2$ let $[a, b] = \{\theta \in \Sigma_2; a \leq \theta \leq b\}$ be the closed interval defined by the

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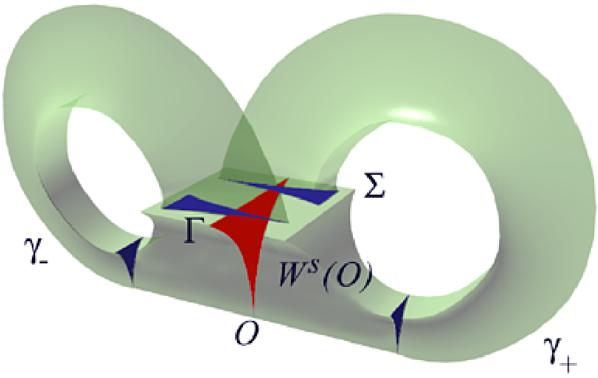

Figure 1. Geometric Lorenz attractor

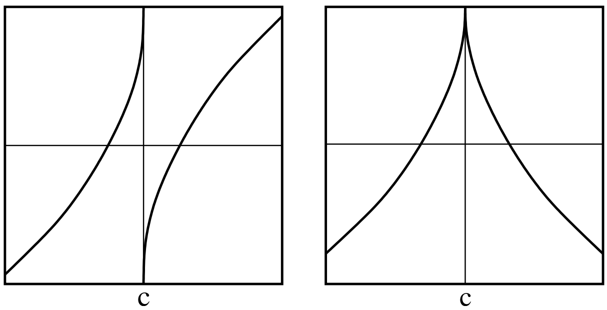

Figure 2. One-dimensional return map

lexicographical order in Σ_2. Let Min_2 (resp. Max_2) denote the set of minimal (resp. maximal) sequences in Σ_0 (resp. Σ_1). We define the Lexicographical world (see [11, 12] and [13]) as $LW = \{(a, b) \in \mathrm{Min}_2 \times \mathrm{Max}_2, \{a, b\} \subset \Sigma[a, b]\}$ where

$$\Sigma[a, b] = \bigcap_{i=0}^{\infty} \sigma^{-i}([a, 0b] \cup [1a, b])$$
$$= \{\theta \in \Sigma_2 : \sigma^i(\theta) \in [a, 0b] \cup [1a, b] \text{ for all } i \in \mathbb{N}_0\}.$$

The ordered subshifts mentioned above are, precisely, those of the form $\Sigma[a, b]$. In fact, since there are uncountably many different $(a, b) \in LW$ then there are uncountable many geometric Lorenz flows, as Guckenheimer and William showed.

We observe that these ordered subshifts not only modelled Geometric Lorenz flows. In fact, let $0 < c < 1$ and DC_c be the set of those maps $f : ([0, 1] \setminus \{c\}) \to [0, 1]$ such that

1. $f|_{[0,c[}$ and $f|_{]c,1]}$ are continuous increasing maps,
2. $f(c^-) = 1$ and $f(c^+) = 0$.

For $f \in \mathrm{DC}_c$ let

$$\Lambda_f = \left([0,1] \setminus \bigcup_{n=0}^{\infty} f^{-n}(c)\right) \subset [0,1]$$

the "continuity" set of the map f. Associated to any $x \in \Lambda_f$ (see section 2), there is an itinerary $I_f(x) \in \Sigma_2$ such that $\sigma \circ I_f(x) = I_f \circ f(x)$ for any $x \in \Lambda_f$. These itineraries allow us to define $I_f(x^{\pm})$, for any $x \in [0,1]$ and the set $J_f = \{I_f(x^{\pm}) : x \in [0,1]\}$. It is not hard to prove that $J_f = \Sigma([a_f, b_f])$ where $a_f = I_f(0^+)$ and $b_f = I_f(1^-)$ are the *kneading* sequences associated to the map f. So the dynamic of the ordered subshift $\sigma\colon \Sigma[a_f, b_f] \to \Sigma[a_f, b_f]$ essentially represents the dynamic of any $f \in \mathrm{DC}_c$ any $0 < c < 1$.

For elements in DC_c there are (at least) three degrees of complexity for the dynamics of its elements that are:

(C_1) $\mathrm{DC}_c(1) = \{f \in \mathrm{DC}_c : f(0) > f(1)\}$;
(C_2) $\mathrm{DC}_c(2) = \{f \in \mathrm{DC}_c : f(0) = f(1)\}$ and
(C_3) $\mathrm{DC}_c(3) = \{f \in \mathrm{DC}_c : f(0) < f(1)\}$.

Maps in C_1 and C_2 may have the same combinatorial dynamics but it is certainly different from the combinatorial dynamics that presents the elements in C_3. The difference, in the combinatorial dynamics, comes from the sequences $(a_f, b_f) \in LW$ that elements in $\mathrm{DC}_c(1) \cup \mathrm{DC}_c(2)$ and $\mathrm{DC}_c(3)$ may attach.

In the present work we present a classification of all the possible sequences $a_f \in \mathrm{Min}_2$ that can be attached by the elements in $\mathrm{DC}_c(1) \cup \mathrm{DC}_c(2)$. Clearly, a similar result is true for the sequences $b_f \in \mathrm{Max}_2$ as we will make clear at the end of the work.

In a forthcoming paper(see [16]) we will present a classification of all possible sequences $a_f \in \mathrm{Min}_2$ that can be attached by the elements in $\mathrm{DC}_c(3)$.

2. Statement of the Main result

For $\theta \in \Sigma_2$ and $k \in \mathbb{N}$ such that $\theta_{k+j} = \theta_j$, for $j \in \mathbb{N}$ we will write $\theta = \underline{\theta_0 \cdots \theta_{k-1}}$. If $a = \alpha_0 \cdots \alpha_{k-1} a_k$ we will denote $a_* = \alpha_0 \cdots \alpha_{k-1} b_k$ with $b_k \neq a_k$.

Let $\underline{a_1}$, $\underline{a_2}$ be two periodic sequences in Σ_2. The sequence $m(a_1, a_2) = \underline{a_1 a_2}$ will be called the *average* of the sequences $\underline{a_1}$ and $\underline{a_2}$. For a string $a = a_0 \cdots a_k$, $k \geq 0$ we will denote by a^n the string $a \cdots a$ (n times). For a subset $A \subset \Sigma_2$ whose points are isolated we will say that $a_1, a_2 \in A$ are *consecutive* if $a_1 < a_2$ and there is not $a \in A$ such that $a_1 < a < a_2$.

Examples. For $\underline{a_1} = \underline{01}$, $\underline{a_2} = \underline{011}$ we have $m(a_1, a_2) = \underline{01011}$ and $\underline{a_1}$ and $\underline{a_2}$ are consecutive sequences in the set

$$A_0 = \{\underline{0^n 1}, \underline{01^n} : n \in \mathbb{N}\} = \{\ldots, \underline{0001}, \underline{001}, \underline{01}, \underline{011}, \underline{0111}, \ldots\}$$

Let us consider A_0 as in the example and define

$$A_{n+1} = A_n \cup \{m(a_1, a_2) : a_1, a_2 \in A_n \text{ and } \underline{a_1} < \underline{a_2} \text{ are consecutives}\}$$

for any $n \geq 0$.

Let $A_\infty = \bigcup_{n=0}^{\infty} A_n$. The aim of the present work is to prove

THEOREM (2.1). *The set* $\mathrm{KDC}_c = \{a_f \in \Sigma_0 : f \in (\mathrm{DC}_c(2) \cup \mathrm{DC}_c(1))\} = \overline{A_\infty}$.

It is not hard to see that for any $R_\alpha\colon [0,1] \to [0,1]$, $0<\alpha<1$ such that $R_\alpha(x) = x+\alpha$ if $0 \le x < 1-\alpha$ and $R_\alpha(x) = x+\alpha-1$ if $1-\alpha < x \le 1$, there is $f_\alpha \in \mathrm{DC}_c(2)$ and an homeomorphism $H\colon [0,1] \to [0,1]$ such that $H(0)=0$, $H(1-\alpha)=c$, $H(1)=1$ and $H \circ R_\alpha \circ H^{-1}(x) = f_\alpha(x)$. Clearly the rotation number of R_α (when considered as a map $\mathbb{S}^1 \to \mathbb{S}^1$) is α, and consequently the rotation number of f_α, is α. It is not hard to see that for $a_\alpha = a_{f_\alpha}$ and for $m(n) = \#\,\{\text{number one presents in } a_0 \cdots a_{n-1}\}$ that

$$\lim_{n\to\infty} \frac{m(n)}{n} = \alpha.$$

For instance, for the canonical family $f : \mathbb{S}^1 \to \mathbb{S}^1$ given by $f_{\tau,b}(x) = (x+\tau+b\sin(2\pi x)) \mod 1$, it is well known that for $b \le 1$ the map $f_{\tau,b}$ is an orientation preserving homeomorphism of the circle and a description of the set $T_r = \{(\tau,b) : \text{rotation number}(f_{\tau,b}) = r\} \subset \{(\tau,b) : \tau \le 1\}$ is known. For rational r these sets are known as the "Arnold tongue of rational number r" (see [1]) and for r irrational these sets are curves with the following remarkable property: for fixed b the set $T_r \cap \{(\tau,b) : 0 \le \tau \le 1\}$ has positive Lebesgue measure (see [8]). This canonical family is homologous to a family $\tilde{f}_{\tau,b} \in \mathrm{DC}_c(2)$ and the two parameter family of sequences $a(\tau,b) = a_{\tilde{f}_{\tau,b}}$ can be used, instead of the rotation number, to parameterize the bifurcation diagram associated to the canonical family $(f_{\tau,b})$ such that $0<\tau<1$, $0 \le b \le 1$ (compare with [3] and [5]).

The good news is that this parametrization for the bifurcation diagram, for two parameter families of elements in $\mathrm{DC}_c(2)$, can be extended to parameterized families in $\mathrm{DC}_c(2) \cup \mathrm{DC}_c(3)$ as it is shown in [11] and [12].We will present the exact scope of this extension in a forthcoming work (see [16]).

Remark: The search for the characterization of the kneading sequences of maps in DC_c in not new. In fact, it has a long history as may be appreciated in [18] and the references there in. Nevertheless, our approach is certainly new.

For instance, some differences between our approach and those described in [18] include the followings: (a) our presentation of the set of itineraries associated to elements in $\mathrm{DC}_c(2)$ is simpler than the representation given in [18]; (b) our presentation is more explicit (with respect to the sequences that effectively belongs to KDC_c) and (c) our presentation applies to elements in $\mathrm{DC}_c(3)$ (as we will shown in a forth coming paper, see [16])

The sequence of this paper is organized as follows: In section 3 we introduce the lexicographical world, describe the set DC_c and we prove the realization lemma for elements in $DC_c(2)$. In section 4 prove some properties of the set A_∞ and in section 5 we will prove our main result.

3. Symbolic dynamics for Lorenz maps

(3.1) The set DC_c. In the sequel DC_c will denote the set of maps $f : ([0,1] \setminus \{c\}) \to [0,1]$ such that

1. $f_{[0,c[}$ and $f_{]c,1]}$ are continuous and increasing maps,
2. $f(c^-) = \lim_{x\uparrow c} f(x) = 1$, $f(c^+) = \lim_{x\downarrow c} f(x) = 0$.

For elements $f, g \in \mathrm{DC}_c$ we can define $d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,c[\} + \sup\{|f(x)-g(x)| : x \in]c,1]\}$. Since $f|_{[0,c[}$ and $g|_{[0,c[}$ (resp. $f|_{]c,1]}$ and

$g|_{]c,1]}$) can be extended (in a unique way) to continuous increasing maps $\bar{f}$, $\bar{g}\colon [0,c] \to [0,1]$ (resp. $\tilde{f}, \tilde{g}\colon [c,1] \to [0,1]$) such that

1. $\bar{f}(c) = \bar{g}(c) = 1$ (resp. $\tilde{f}(c) = \tilde{g}(c) = 0$) and
2. $\sup\{|\bar{f}(x) - \bar{g}(x)| : x \in [0,c]\} = \sup\{|f(x) - g(x)| : x \in [0,c[\}$ (resp. $\sup\{|\tilde{f}(x) - \tilde{g}(x)| : x \in [c,1]\} = \sup\{|f(x) - g(x)| : x \in]c,1]\}$)

then we conclude that $d\colon \mathrm{DC}_c \times \mathrm{DC}_c \to [0,1]$ is a metric and (DC_c, d) is a complete metric space.

(3.2) The Lexicographical order. Let Σ_2 denote the set of sequences $\theta : \mathbb{N} \to \{0,1\}$ endowed with the topology given by the metric $d(\alpha,\beta) = \frac{1}{2^n}$ where $n = \min\{k : \alpha_k \neq \beta_k\}$. Let $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ be the shift map $\sigma(\theta_0, \theta_1, \theta_2, \ldots) = (\theta_1, \theta_2, \ldots)$. Let Σ_0 and Σ_1 denote the sets $\{\theta \in \Sigma_2 : \theta_0 = 0\}$ and $\{\theta \in \Sigma_2 : \theta_0 = 1\}$, respectively. It is clear that $\Sigma_2 = \Sigma_0 \cup \Sigma_1$ and that the restriction $\sigma|_{\Sigma_i} : \Sigma_i \to \Sigma_2$, $i = 0, 1$ is an homeomorphism. In Σ_2 we consider the *lexicographical order* $\theta < \alpha$ for any $\theta \in \Sigma_0$ and $\alpha \in \Sigma_1$ or $\theta < \alpha$ if there is $n \in \mathbb{N}$ such that $\theta_i = \alpha_i$ for $i = 0, 1, 2, \ldots, n-1$ and $\theta_n = 0$ and $\alpha_n = 1$.

For $\alpha, \beta \in \Sigma_2$ we define $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$. In this situation, $\alpha \leq \beta$, we define

$$\Sigma[\alpha,\beta] = \{\theta \in \Sigma_2 : \alpha \leq \sigma^i(\theta) \leq \beta \text{ for all } i \in \mathbb{N}_0\} = \bigcap_{n=0}^{\infty} \sigma^{-n}([\alpha, 0\beta] \cup [1\alpha, \beta]).$$

(3.3) The set Σ_{a_f,b_f}. For $f \in \mathrm{DC}_c$ let

$$\Gamma_f = \left([0,1] \setminus \bigcup_{j=0}^{\infty} f^{-j}(\{c\}) \right)$$

denote the set of "continuity" of the map. For $x \in \Gamma_f$ we define $I_f(x) \in \Sigma_2$ by $I_f(x)(i) = 0$ if $f^i(x) < c$ and $I_f(x)(i) = 1$ if $f^i(x) > c$.

For $x = c$ we define $I_f(c^-) = \lim_{x \uparrow c, x \in \Gamma_f} I_f(x)$ and $I_f(c^+) = \lim\limits_{x \downarrow c, x \in \Gamma_f} I_f(x)$.

In the same way to any $x \in \bigcup_{j=0}^{\infty} f^{-j}(\{c\})$ such that $f^i(x) \neq c$, $0 \leq i < n$ and $f^n(x) = c$ we associate the sequences $I_f(x^{\pm}) = (I_f(x)(0), \ldots, I_f(x)(n-1), I_f(c^{\pm}))$ where $I_f(x)(i) = 0$ if $f^i(x) < c$ and $I_f(x)(i) = 1$ if $f^i(x) > c$ for $0 \leq i < n$.

For $x \in \Gamma_f$ we define $I_f(x^{\pm}) = I_f(x)$. Let $I_f = \{I_f(x^{\pm}) : x \in [0,1]\}$ and let us denote by $a_f = I_f(0^+)$, $b_f = I_f(1^-)$. The following lemma is a classical fact that associate an ordered symbolic dynamical system to a Lorenz map on the interval via kneading sequences. See, for instance [10] or [13].

LEMMA (3.1). $I_f = \Sigma[a_f, b_f]$

Let $f, g \in \mathrm{DC}_c$. We will say that f has *essentially* the same dynamics as g if $I_f = I_g$. We observe that, in this situation, up to the existence of some intervals where the itineraries of the points are the same, the dynamics of the maps f and g are topologically equivalent (see [4] and [10]).

(3.4) The Lexicographical world. Let $\mathrm{Min}_2 = \{a \in \Sigma_0 : \sigma^k(a) \geq a \text{ for all } k \in \mathbb{N}\}$ and $\mathrm{Max}_2 = \{b \in \Sigma_1 : \sigma^k(b) \leq b \text{ for all } k \in \mathbb{N}\}$. The elements in Min_2 (resp. Max_2) will be called *minimal* (resp. *maximal*).

Notes:

1. Min_2 and Max_2 are closed sets in Σ_2.
2. Assume $a \in \mathrm{Min}_2 \cap \Sigma_0$ is a periodic sequence with period $a_0 \cdots a_k$ then for $k \geq 1$ we have $a_0 = 0$ and $a_k = 1$.
3. Assume $b \in \mathrm{Max}_2 \cap \Sigma_1$ is a periodic sequence with period $b_0 \cdots b_k$ then for $k \geq 1$ we have $b_0 = 1$ and $b_k = 0$.

The set $LW = \{(a, b) \in \mathrm{Min}_2 \times \mathrm{Max}_2 : \{a, b\} \subset \Sigma[a, b]\}$ will be called the *lexicographical world.*

(3.5) The realization lemma. Let us consider $(a, b) \in LW$. The following result was proved in [10] and [13].

PROPOSITION (3.2). *There is $f \in \mathrm{DC}_c$ such that $I_f = \Sigma[a, b]$.*

In this work we will prove the following

PROPOSITION (3.3). *For any $a \in A_\infty$ there is an element $f \in \mathrm{DC}_c(2)$ such that*

1. *$f|_{[0,c[}$ and $f|_{]c,1]}$ are injective,*
2. *$a_f = a$ and $b_f = \max\{\sigma^i(a) : i \in \mathbb{N}\}$.*

That is, any element in A_∞ is realized as the a_f–kneading sequence of some element $f \in \mathrm{DC}_c(2)$ that induces an homeomorphism $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.

Proof. Let $a_0 < a_1 < \cdots < a_{\text{last}} = a_p < 0\underline{1} < 1\underline{0} < b_{\text{first}} = b_{p+1} < \cdots < b_k = b(a) = \max\{\sigma^i(a) : i \in \mathbb{N}\}$ denote consecutive elements in the finite orbit $\{\sigma^j(a) : j \in \mathbb{N}\} = \{a, \sigma(a), \sigma^2(a), \dots, \sigma^{k-1}(a)\}$.

Consider closed intervals $L_0, L_1, \dots, L_k$ such that

$$\bigcup_{j=0}^{k} L_j = [0, 1], \min(L_0) = 0,$$
$$\max(L_0) = \min(L_1), \dots, \max(L_p) = c = \min(L_{p+1}),$$
$$\max(L_{p+1}) = \min(L_{p+2}), \dots, \max(L_{k-1}) = \min(L_k) \quad \text{and}$$
$$\max(L_k) = 1.$$

Here $\min[x, y] = x$ and $\max[x, y] = y$.

Now define a map f which is continuous and strictly increasing and satisfies:

$$f(L_{p+1}) = L_0, f(L_{p+2}) = L_1, \dots, f(L_k) = L_{k-p-1}, f(L_0) = L_{k-p},$$
$$f(L_1) = L_{k-p+1}, \dots, f(L_p) = L_k.$$

Clearly $f \in \mathrm{DC}_c(2)$ satisfy 1 and 2. □

COROLLARY (3.4). *Let $a \in \left(\overline{A_\infty} \setminus A_\infty\right)$ then there is a map $f \in \mathrm{DC}_c(2)$ that satisfies*

1. *$f|_{[0,c[}$ and $f|_{]c,1]}$ are injective;*
2. *$a_f = a$ and $b_f = b(a) = \sup\{\sigma^i(a) : i \in \mathbb{N}\}$.*

Proof. Take a sequence $(a_n) \subset A_\infty$ such that $\lim_{n\to\infty} a_n = a$. Let $(f_n) \subset \mathrm{DC}_c(2)$ the sequence constructed as in the proposition. It is clear that this construction can be realized in such way that the extension $(\bar{f}_n)$, $\bar{f}_n : [0, c] \longrightarrow [0, 1]$ and

$\tilde{f}_n : [c, 1] \longrightarrow [0, 1]$ form a Cauchy sequence. Its limits in $\mathrm{DC}_c(2)$ satisfies 1 and 2. □

As a consequence of proposition 3.3 and its corollary we have

COROLLARY (3.5). $\overline{A_\infty} \subset \mathrm{KDC}_c$.

(3.6) The renormalization map. Let $a = a_0 \cdots a_k$ and $b = b_0 \cdots b_p$ be two different strings of 0's and 1's such that $\underline{a} < \underline{b}$. Let $\Sigma(a, b)$ denote the set of sequences $\theta : \mathbb{N} \longrightarrow \{a, b\}$ with the induced topology as a subset of Σ_2.

The *renormalization map* $R_{a,b} : \Sigma_2 \to \Sigma(a, b)$ is defined by $R_{a,b}(c_0, c_1, c_2, \ldots) = (\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots)$ where $\bar{c}_i = a$ if $c_i = 0$ and $\bar{c}_i = b$ if $c_i = 1$

Examples. $R_{a,b}(\underline{01}) = \underline{ab}$, $R_{01,1}(\underline{011}) = \underline{0111}$, $R_{0,01}(\underline{011}) = \underline{00101}$.

It is clear that the renormalization map is continuous and bijective.

LEMMA (3.6). *Assume $\alpha \leq \beta$ in Σ_2 then $R_{a,b}(\alpha) \leq R_{a,b}(\beta)$ in Σ_2. (That is: the renormalization map is order preserving.)*

Proof. If length$(a) = k = p =$length(b) then the result is obvious. In fact for some $r < k = p$ we have $a_i = b_r$, $0 \leq i < r$ and $a_r = 0, b_r = 1$. Since $\alpha \leq \beta$ then there is n_0 such that $\alpha_i = \beta_i$ for $0 \leq i < n_0$ and $\alpha_{n_0} = 0, \beta_{n_0} = 1$. So $R_{a,b}(\alpha) = (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{n_0-1}, a, \ldots) < R_{a,b}(\beta) = (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{n_0-1}, b, \ldots)$.

Assume length$(a) = k < p =$length(b). If for some $r \leq k$ we have $a_r \neq b_r$ then the result follows as in the previous case.

So, let us assume that $b = a^s b_{sk}, \ldots, b_p$ with $0 \leq p - sk < k$. If $b_{sk+i} \neq a_i$ for some $0 \leq i < p - sk$ then the result follows as in the previous case.

Assume $b_{sk+i} = a_i$, $0 \leq i \leq p - sk$. In this condition, and because $\underline{a} < \underline{b}$, we must have $a_0 \cdots a_{p-sk} a_0 \cdots a_{(s+1)k-p-1} > a_0 \cdots a_k$ and, consequently, $ba > a^{s+1}$ which imply $ba > ab$. So, we have $\underline{a} < a\underline{b} < b\underline{a} < \underline{b}$. Inductively, for any $\theta_0\theta_1 \cdots \theta_k$ with $\theta_i \in \{a, b\}$ we have $\theta_0 \cdots \theta_k\underline{a} < \theta_0 \cdots \theta_k a\underline{b} < \theta_0 \cdots \theta_k b\underline{a} < \theta_0 \cdots \theta_k\underline{b}$.

Now, any $\alpha = \alpha_0 \cdots \alpha_k 0 \alpha_{k+2} \cdots$ satisfy

$$R_{a,b}(\alpha) = \bar{\alpha}_0 \cdots \bar{\alpha}_k a \bar{\alpha}_{k+2} \cdots \in [\bar{\alpha}_0 \cdots \bar{\alpha}_k\underline{a}, \bar{\alpha}_0 \cdots \bar{\alpha}_k a\underline{b}]$$

and any

$$\beta = \alpha_0 \cdots \alpha_k 1 \beta_{k+2} \cdots$$

satisfy

$$R_{a,b}(\beta) = \bar{\alpha}_0 \cdots \bar{\alpha}_k b \bar{\beta}_{k+2} \cdots \in [\bar{\alpha}_0 \cdots \bar{\alpha}_k b\underline{a}, \bar{\alpha}_0 \cdots \bar{\alpha}_k\underline{b}].$$

In particular, $R_{a,b}(\alpha) < R_{a,b}(\beta)$.

The case length$(a) = k >$length$(b) = p$ can be done in a similar way. □

LEMMA (3.7). *Assume $a \in \mathrm{Min}_2$ then $\{R_{0,01}(a), R_{01,1}(a)\} \subset \mathrm{Min}_2$. That is $R_{0,01}(\mathrm{Min}_2) \subset \mathrm{Min}_2$ and $R_{01,1}(\mathrm{Min}_2) \subset \mathrm{Min}_2$.*

Proof. Assume $a \in \mathrm{Min}_2$ is a periodic sequence. That is

$$a = \underline{0^{k_1}1^{p_1}0^{k_2}1^{p_2} \cdots 0^{k_r}1^{p_r}}.$$

Since $\sigma^i(a) \geq a$ for all $i \in \mathbb{N}$ we have $\underline{0^{k_j}1^{p_j}0^{k_{j+1}}1^{p_{j+1}} \cdots 0^{k_{j-1}}1^{p_{j-1}}} \geq a$ for any $j = 1, 2, \ldots, r$. In particular $R_{0,01}(\underline{0^{k_j}1^{p_j} \cdots 0^{k_{j-1}}1^{p_{j-1}}}) = \underline{0^{k_j}(01)^{p_j} \cdots 0^{k_{j-1}}(01)^{p_{j-1}}} \geq R_{0,01}(a) = \underline{0^{k_1}(01)^{p_1}0^{k_2}(01)^{p_2} \cdots 0^{k_r}(01)^{p_r}}$ for any $j = 1, 2, \ldots, r$. Since any other

$\sigma-$iteration of $R_{0,01}(a)$ start with (01) or 1 we have $\sigma^k(R_{0,01}(a)) \geq R_{0,01}(a)$ for all $k \in \mathbb{N}$.

In a similar way we obtain the result for $R_{01,1}$. □

4. Some properties of A_∞

Let $a = a_0 \cdots a_k$ and $b = b_0 \cdots b_p$ be two strings of zeros and ones. We define the average of these two strings as the sequence $m(a,b) = \underline{ab} = \overline{a_0 \cdots a_k b_0 \cdots b_p}$, which is a periodic sequence.

LEMMA (4.1). *If $a = a_0 \cdots a_k$, $b = b_0 \cdots b_p$ are two strings such that $\underline{a} \in \mathrm{Min}_2$ and $\underline{b} \in \mathrm{Min}_2$ and $\underline{a} < \underline{b}$ then $m(a,b) \in \mathrm{Min}_2$ and $\underline{a} < m(a,b) < \underline{b}$.*

Proof. Let us first prove the inequality. Assume $\underline{a} = \overline{a_0 \ldots a_k}$ and $\underline{b} = \overline{b_0 \ldots b_k}$, then the condition $\underline{a} < \underline{b}$ implies that there exists $i,\ 0 < i < k$ such that $a_i = 0$ and $b_i = 1$.So, in this situation we easily get $\underline{a} < m(a,b) < \underline{b}$.

Let us now consider that $\underline{a} = \overline{(0^{k_1}1^{l_1})^{r_1}(0^{k_2}1^{l_2})^{r_2} \ldots (0^{k_s}1^{l_s})^{r_s}}$. The condition $a \in \mathrm{Min}_2$ imply that $k_1 \leq k_2$ and $k_1 = \min\{k_1, k_2, \ldots, k_s\}$. Let us assume that *period*$(\underline{b}) >$ *period*$(\underline{a})$. Let us write $\underline{b} = \overline{b_0 b_1 \ldots b_p}$ and $\underline{a} = \alpha_0\alpha_1 \ldots \alpha_p \ldots$.

Claim. In this situation there is $i,\ 1 \leq i \leq p$ such that $b_i \neq \alpha_i$.

In fact, otherwise $b_0 = \alpha_0,\ b_1 = \alpha_1, \ldots, b_p = \alpha_p$. hence, and without loss, we can assume that $\underline{b} = \overline{((0^{k_1}1^{l_1})^{r_1}(0^{k_2}1^{l_2})^{r_2} \ldots (0^{k_s}1^{l_s})^{r_s})^t 0^{k_1}1^{p_1}}$. Let $N = [(k_1 + p_1)r_1 + (k_2 + p_2)r_2 \cdots + (k_s + p_s)r_s]t$ then we have

$$\sigma^N(\underline{b}) = 0^{k_1}1^{p_1}((0^{k_1}1^{l_1})^{r_1}(0^{k_2}1^{l_2})^{r_2} \ldots (0^{k_s}1^{l_s})^{r_s})^t 0^{k_1}1^{p_1} \ldots$$

So, we must have $\sigma^N(\underline{b}) < \underline{b}$ a contradiction with $\underline{b} \in \mathrm{Min}_2$. Therefore, in this case, we obtain the claim.

In a similar way, for $\underline{b} = \overline{((0^{k_1}1^{l_1})^{r_1}(0^{k_2}1^{l_2})^{r_2} \ldots (0^{k_s}1^{l_s})^{r_s})^t 0^{k_1}1^{i}}$ for some $1 \leq i \leq r_1$ or $\underline{b} = \overline{((0^{k_1}1^{l_1})^{r_1}(0^{k_2}1^{l_2})^{r_2} \ldots (0^{k_s}1^{l_s})^{r_s})^t (0^{k_1}1^{p_1})^{i}}$ for some $1 \leq i \leq r_2$ or $\underline{b} = \overline{((0^{k_1}1^{l_1})^{r_1}(0^{k_2}1^{l_2})^{r_2} \ldots (0^{k_s}1^{l_s})^{r_s})^t (0^{k_1}1^{p_1})^{r_1}(0^{k_2}1^{p_2})^{i}}$ for some $1 \leq i \leq r_2$, we obtain a contradiction with $\underline{b} \in \mathrm{Min}_2$.

Now, by the claim, we must have

$$\underline{a} = \alpha_0 \ldots \alpha_{i-1}0 \ldots \alpha_p \cdots = a_0 a_1 \ldots a_k a_0 \ldots a_k \ldots$$

and

$$\underline{b} = \alpha_0\alpha_1 \ldots \alpha_{i-1}1 \ldots b_p \ldots .$$

Assuming $i = (k+1)p + l$ then $a_{l-1} = 0$ and $\underline{b} = (a_0 \ldots a_k)^p a_0 \ldots a_{l-1}1$. This implies that

$$\underline{a} = (a_0 \ldots a_k)^{p+1} a_0 \ldots a_{l-1} a_l \cdots < \underline{ab} = (a_0 \ldots a_k)^{p+1} a_0 \ldots a_{l-1}1 \text{ and } \underline{ab} < \underline{b}$$

as we announced.

Let us now prove that $m(a,b) \in \mathrm{Min}_2$. So, assume $\underline{a} = \overline{0^{k_1}1^{p_1}0^{k_2}1^{p_2} \cdots 0^{k_s}1^{p_s}}$ and $\underline{b} = \overline{0^{l_1}1^{m_1}0^{l_2}1^{m_2} \cdots 0^{l_t}1^{m_t}}$. Without loss of generality let us assume that $s \geq t$.

Since $\underline{a}, \underline{b} \in \mathrm{Min}_2$ we have $k_1 = \max\{k_1, \ldots, k_s\}$ and $l_1 = \max\{l_1, \ldots, l_t\}$. Moreover the condition $\underline{a} < \underline{b}$ imply $l_1 \leq k_1$.

We have $\underline{ab} = \overline{0^{k_1}1^{p_1} \cdots 0^{k_s}1^{p_s}0^{l_1}1^{m_1} \cdots 0^{l_t}1^{m_t}}$.

For $1 \leq i \leq t$ put $k_{s+i} = l_i$ and $p_{s+i} = m_i$.

We have $\underline{ab} = \overline{0^{k_1}1^{m_1} \cdots 0^{k_s}1^{m_s}0^{k_{s+1}}1^{p_{s+1}} \cdots 0^{k_{s+t}}1^{p_{s+t}}}$.

Note that if for some j, $1 \le j \le s+t$ we have $\underline{0^{k_j}1^{p_j}0^{k_{j+1}}1^{p_{j+1}}\dots 0^{k_{s+t}}1^{p_{s+t}}0^{k_1}1^{p_1}\dots 0^{k_{j-1}}1^{p_{j-1}}} < \underline{ab}$ then we must have either

1. $k_1 < k_j$ for some $j \in \{2,3,\dots,s+t\}$ and we contradicts $\underline{a} \in \mathrm{Min}_2\,(2 \le j \le s)$ or $\underline{b} \in \mathrm{Min}_2\,(s+1 \le j \le s+t)$ or
2. $k_1 = k_j$ for any $j \in \{2,\dots,s+t\}$. In this situation $p_1 > p_j$ for some $j \in \{2,\dots,s+t\}$. Hence we contradicts $\underline{a} \in \mathrm{Min}_2\,(2 \le j \le s)$ or $\underline{b} \in \mathrm{Min}_2\,(s+1 \le j \le s+t)$.

Therefore: $\sigma^i(\underline{ab}) \ge \underline{ab}$ for any $i \in \mathbb{N}$. □

Now, as in section 2, let $A_0 = \{\underline{0^n1}, \underline{01^n} : n \in \mathbb{N}\}$, $A_{n+1} = A_n \cup \{m(a_1,a_2) : a_1, a_2 \in A_n$ are consecutive sequences$\}$ and $A_\infty = \bigcup_{n=0}^{\infty} A_n$.

COROLLARY (4.2). $A_\infty \subset \mathrm{Min}_2$.

Proof. Clearly $A_0 \subset \mathrm{Min}_2$. Assume $A_n \subset \mathrm{Min}_2$. For $a \in A_{n+1}$ we have either $a \in A_n$ or there are two consecutive sequences $\underline{a_1} < \underline{a_2}$ such that $a = m(a_1,a_2)$. In the first case $a \in \mathrm{Min}_2$ and in the second one $a \in \mathrm{Min}_2$ by lemma 4.1. □

In order to prove the main result, we will give a different presentation of the set A_∞, for this we will construct inductively a set $\mathcal{A}_\infty$ such that: $(\mathcal{A}_\infty \backslash \{\underline{0}, \underline{1}\}) = A_\infty$.

To do so, let us define $\mathcal{A}_0 = \{\underline{0}, \underline{1}\}$, $\mathcal{A}_1 = R_{0,01}(\mathcal{A}_0) \cup R_{01,1}(\mathcal{A}_0)$ and $\mathcal{A}_{n+1} = R_{0,01}(\mathcal{A}_n) \cup R_{01,1}(\mathcal{A}_n)$ for $n \ge 1$.

Examples: $\mathcal{A}_1 = \{\underline{0}, \underline{01}, \underline{1}\}$, $\mathcal{A}_2 = \{\underline{0}, \underline{001}, \underline{01}, \underline{011}, \underline{1}\}$,
$\mathcal{A}_3 = \{\underline{0}, \underline{0001}, \underline{001}, \underline{00101}, \underline{01}, \underline{01011}, \underline{011}, \underline{0111}, \underline{1}\}$

In general we note that $\#(\mathcal{A}_{n+1}) = 2\#(\mathcal{A}_n) - 1 = 2(2^n+1) - 1 = 2^{n+1} - 1$ and

$$\mathcal{A}_n = \{A^{l_{k+1}}B^{l_k}\cdots A^{l_2}B^{l_1}(0), B^{l_{k+1}}A^{l_k}\cdots B^{l_2}A^{l_1}(0), \\ A^{l_{k+1}}B^{l_k}\cdots A^{l_2}B^{l_1}(1), B^{l_{k+1}}A^{l_k}\cdots B^{l_2}A^{l_1}(1)\}$$

for $l_1, l_2, \dots, l_{k+1}$ such that $l_1 + \cdots + l_{k+1} = n$ and $A = R_{0,01}$, $B = R_{01,1}$

PROPOSITION (4.3). *For* $\mathcal{A}_\infty = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ *we have* $A_\infty = (\mathcal{A}_\infty \setminus \{\underline{0}, \underline{1}\})$.

Proof. Let us initially prove that $A_\infty \subset \mathcal{A}_\infty$. In the sequel we will use the notation $01 = A(1) = B(0)$.

For $\underline{0^k1} \in A_0$ we have $\underline{0^k1} = \underline{0^{k-1}01} = \underline{A(0^{k-1}1)} = \underline{A^{k-1}(01)} = \underline{A^k(1)}$.

For $\underline{01^k} \in A_0$ we have $\underline{01^k} = \underline{011^{k-1}} = \underline{B(01^{k-1})} = \underline{B^{k-1}(01)} = \underline{B^k(0)}$. Hence, we get $A_0 \subset \mathcal{A}_\infty$.

Without loss of generality, let us show that $A_n \cap]\underline{A(01)}, \underline{01}[\subset \mathcal{A}_\infty$ for any $n \ge 1$.

In fact, $A_1 \cap]\underline{A(01)}, \underline{01}[= \{\underline{A(01)01}\} = \{\underline{AB(01)}\} \subset \mathcal{A}_\infty$ and we have $A_1 \cap [\underline{A(01)}, \underline{01}] = \{\underline{A(01)}, \underline{01}, \underline{AB(01)}\} \subset \mathcal{A}_\infty$

For $(A_2 \backslash A_1)$ we have $(A_2 \backslash A_1) \cap]\underline{A(01)}, \underline{01}[= \{\underline{A(01)AB(01)}, \underline{AB(01)01}\} = \{\underline{A(01B(01))}, \underline{0010101}\} = \{\underline{ABA(01)}, \underline{ABB(01)}\}$. So, we get $A_2 \cap [\underline{A(01)}, \underline{01}] = \{\underline{A(01)}, \underline{01}, \underline{AB(01)}, \underline{ABA(01)}, \underline{ABB(01)}\}$.

For $(A_3 \backslash A_2)$ we have

$$\begin{aligned}(A_3\backslash A_2)\cap]\underline{A(01)}, \underline{01}[&= \{\underline{A(01)ABA(01)}, \underline{ABA(01)AB(01)}, \underline{AB(01)ABB(01)},\\ &\quad \underline{ABB(01)01}\}\\ &= \{\underline{ABAA(01)}, \underline{ABAB(01)}, \underline{ABBA(01)}, \underline{AB^3(01)}\}.\end{aligned}$$

So, we get

$$\begin{aligned}A_3 \cap [\underline{A(01)}, \underline{01}] = \{&\underline{A(01)}, \underline{01}, \underline{AB(01)}, \underline{ABA(01)}, \underline{ABB(01)}, \underline{ABA^2(01)},\\ &\underline{ABAB(01)}, \underline{ABBA(01)}, \underline{AB^3(01)}\}.\end{aligned}$$

Hence, for $n \geq 3$ let us assume inductively that

$$\begin{aligned}A_n \cap [\underline{A(01)}, \underline{01}] = \{&\underline{A(01)}, \underline{01}, \underline{AB(01)}, \underline{ABA(01)}, \underline{ABB(01}, \underline{ABA^2(01)},\\ &\underline{ABAB(01)}, \underline{ABBA(01)}, \underline{ABB^2(01)}, \dots, \underline{ABA^{n-1}(01)},\\ &\underline{ABA^{n-2}B(01)}, \underline{ABA^{n-3}BA(01)}, \underline{ABA^{n-3}B^2(01)}, \dots,\\ &\underline{ABB^{n-3}A^2(01)}, \underline{ABB^{n-3}AB(01)}, \underline{ABB^{n-2}A(01)},\\ &\underline{ABB^{n-1}(01)}\} \subset \mathcal{A}_\infty.\end{aligned}$$

In this situation for $(A_{n+1} \backslash A_n)$ we have

$$\begin{aligned}(A_{n+1}\backslash A_n)\cap]\underline{A(01)}, \underline{01}[&= \{\underline{A(01)ABA^{n-1}(01)}, \underline{ABA^{n-1}(01)ABA^{n-2}(01)},\\ &\quad \underline{ABA^{n-2}(01)ABA^{n-2}B(01)},\\ &\quad \underline{ABA^{n-2}B(01)ABA^{n-3}B(01)}, \dots\\ &\quad \underline{ABB^{n-2}A(01)ABB^{n-2}(01)},\\ &\quad \underline{ABB^{n-2}(01)ABB^{n-1}(01)}, \underline{ABB^{n-1}(01)01}\}\\ &= \{\underline{ABA^n(01)}, \underline{ABA^{n-1}B(01)}, \underline{ABA^{n-2}BA(01)},\\ &\quad \underline{ABA^{n-2}B^2(01)}, \dots \underline{ABB^{n-1}A(01)}, \underline{ABB^n(01)}\}.\end{aligned}$$

We conclude that $(A_{n+1}\backslash A_n)\cap]\underline{A(01)}, \underline{01}[\subset \mathcal{A}_\infty$. We proceed in a similar way to show that $A_n \cap [\underline{A^{i+1}(01)}, \underline{A^i(01)}] \subset \mathcal{A}_\infty$ for $i \geq 1$. In this case we get

$$\begin{aligned}A_n \cap [\underline{A^{i+1}(01)}, \underline{A^i(01)}] = \{&\underline{A^{i+1}(01)}, \underline{A^i(01)}, \underline{A^iB(01)}, \underline{A^iBA(01)},\\ &\underline{A^iBB(01)}, \dots, \underline{A^iBA^{n-1}(01)}, \underline{A^iBA^{n-2}B(01)},\\ &\underline{A^iBA^{n-3}BA(01)}, \dots, \underline{A^iBB^{n-2}A(01)},\\ &\underline{A^iBB^{n-1}(01)}\} \subset \mathcal{A}_\infty.\end{aligned}$$

We also can prove, simmilarly, that $A_n \cap [\underline{B^{i-1}(01)}, \underline{B^i(01)}] \subset \mathcal{A}_\infty$ for $i \geq 1$. So we conclude that $A_n \subset \mathcal{A}_\infty$ for all $n \geq 1$ and then $A_\infty \subset \mathcal{A}_\infty$.

Let us now prove that $(\mathcal{A}_\infty \setminus \{\underline{0}, \underline{1}\} \subset A_\infty$. It is clear that

$$\left(\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \setminus \{\underline{0}, \underline{1}\}\right) \subset A_\infty.$$

Let us assume that $\left(\bigcup_{i=0}^{n} \mathcal{A}_i \setminus \{\underline{0}, \underline{1}\}\right) \subset A_\infty$. Let $\alpha \in (\mathcal{A}_{n+1} \setminus \{\underline{0}, \underline{1}\})$ we have that $\alpha = R_{0,01}(a)$ or $\alpha = R_{01,1}(a)$ for some $a \in \mathcal{A}_n$. By the inductive hypothesis we have $a \in A_\infty$.

For $a = \underline{0^m1}$ we have

$$\alpha = R_{01,1}(\underline{0^m1}) = \underline{(01)^m1} = \underline{(01)^{m-1}011} = m(01, (01)^{m-2}011) \in A_\infty$$

or $\alpha = R_{0,01}(\underline{0^m1}) = \underline{0^{m+1}1} \in A_\infty$.

For $a = \underline{01^m}$ we have

$$\alpha = R_{0,01}(\underline{01^m}) = \underline{0(01)^m} = \underline{001(01)^{m-1}} = m(001(01)^{m-1}, 01) \in A_\infty$$

or $\alpha = R_{01,1}(\underline{01^m}) = \underline{011^m} = \underline{01^{m+1}} \in A_\infty$.

In all the other cases there is an m such that **(i)** $\underline{0^{m+1}1} < a < \underline{0^m1}$ or **(ii)** $\underline{01^m} < a < \underline{01^{m+1}}$. Without loss of generality let us assume that **(i)** is the case. In this situation $a \in A_p$ for some $p \in \mathbb{N}$ and $a = m(a_1, a_2)$ for $\underline{a_1}, \underline{a_2} \in A_{p-1}$ two consecutives sequences. We may have **(*)** $\underline{0^{m+1}1} \le \underline{a_1} < \underline{a_2} \le \underline{0^{m+1}10^m1}$ or $(\triangle)\, \underline{0^{m+1}10^m1} \le a_1 < a_2 \le \underline{0^m1}$. Without loss of generality let us assume that **(*)** is the case.

For $\underline{a_1} = \underline{0^m1}$ we must have $\underline{a_2} = \underline{(0^{m+1}1)^{p-1}0^m1}$ and $a = \underline{(0^{m+1}1)^p0^m1}$. In this situation for α we have either

$$\alpha = R_{0,01}\left(\underline{(0^{m+1}1)^p0^m1}\right) = \underline{(0^{m+2}1)^p0^{m+1}1} \in A_p$$

or

$$\begin{aligned}\alpha &= R_{01,1}\left(\underline{(0^{m+1}1)^p0^m1}\right) = \underline{((01)^{m+1}1)^p(01)^m1} \\ &= \underline{((01)^m011)^p(01)^{m-1}011} \in A_\infty.\end{aligned}$$

For $a_1 = \underline{(0^{m+1}1)^{p-1}0^m1}$ we must have $a_2 = \underline{(0^{m+1}1)^{p-2}0^m1}$ and

$$a = \underline{(0^{m+1}1)^{p-1}0^m1(0^{m+1}1)^{p-2}0^m1}.$$

Hence, for α we have either

$$\begin{aligned}\alpha &= R_{0,01}\left(\underline{(0^{m+1}1)^{p-1}0^m1(0^{m+1}1)^{p-2}0^m1}\right) \\ &= \underline{(0^{m+2}1)^{p-1}0^{m+1}1(0^{m+2}1)^{p-2}0^{m+1}1} \\ &= m\left((0^{m+2}1)^{p-1}0^{m+1}1, (0^{m+2}1)^{p-2}0^{m+1}1\right) \in A_p,\end{aligned}$$

since $\underline{(0^{m+2}1)^{p-1}0^{m+1}1}$ and $\underline{(0^{m+2}1)^{p-2}0^{m+1}1}$ are consecutives sequences in A_{p-1} or

$$\begin{aligned}\alpha &= R_{01,1}\left(\underline{(0^{m+1}1)^{p-1}0^m1(0^{m+1}1)^{p-2}0^m1}\right) \\ &= \underline{((01)^{m+1}1)^{p-1}(01)^m1((01)^{m+1}1)^{p-2}(01)^m1} \\ &= \underline{((01)^m011)^{p-1}(01)^{m-1}011((01)^m011)^{p-2}(01)^{m-1}01} \\ &= m(((01)^m011)^{p-1}(01)^{m-1}011, ((01)^m011)^{p-2}(01)^{m-1}011) \in A_\infty.\end{aligned}$$

For $a_1 = \underline{(0^{m+1}1)^{p-2}0^m1}$ we must have $a_2 = \underline{(0^{m+1}1)^{p-2}0^m1(0^{m+1}1)^{p-3}0^m1}$ and $a = \underline{(0^{m+1}1)^{p-2}0^m1(0^{m+1}1)^{p-2}0^m1(0^{m+1}1)^{p-3}0^m1}$. In this case for α we have either

$$\begin{aligned}\alpha &= R_{0,01}(a) = \underline{(0^{m+2}1)^{p-2}0^{m+1}1(0^{m+2}1)^{p-2}0^{m+1}1(0^{m+2}1)^{p-3}0^{m+1}1} \\ &= m((0^{m+2}1)^{p-2}0^{m+1}1, (0^{m+2}1)^{p-2}0^{m+1}1(0^{m+2}1)^{p-3}0^{m+1}1) \in A_\infty\end{aligned}$$

or

$$\begin{aligned}\alpha = R_{01,1}(a) &= \underline{((01)^m 011)^{p-2}(01)^{m-1}011((01)^m 011)^{p-3}(01)^{m-1}011} \\ &= m(((01)^m 011)^{p-2}(01)^{m-1}011, ((01)^m 011)^{p-3}(01)^{m-1}011) \in A_\infty .\end{aligned}$$

For $a_1 = \underline{(0^{m+1}1)^{p-2}0^m 1(0^{m+1}1)^{p-3}0^m 1}$ we must have $a_2 = \underline{(0^{m+1}1)^{p-3}0^m 1}$. So, we get $a = \underline{(0^{m+1}1)^{p-2}0^m 1((0^{m+1}1)^{p-3}0^m 1)^2}$. Therefore, for α we have either

$$\begin{aligned}\alpha = R_{0,01}(a) &= \underline{(0^{m+2}1)^{p-2}0^{m+1}1((0^{m+2}1)^{p-3}0^{m+1}1)^2} \\ &= m((0^{m+2}1)^{p-2}0^{m+1}1(0^{m+2}1)^{p-3}0^{m+1}1, (0^{m+2}1)^{p-3}0^{m+1}1) \in A_\infty\end{aligned}$$

or

$$\begin{aligned}\alpha = R_{01,1}(a) &= \underline{((01)^m 011)^{p-2}(01)^m 011(((01)^m 011)^{p-3}(01)^{m-1}011)^2} \\ &= m(((01)^m 011)^{p-2}(01)^m 011((01)^m 011)^{p-3}(01)^{m-1}011, \\ &\qquad ((01)^m 011)^{p-3}(01)^{m-1}011) \\ &\in A_\infty .\end{aligned}$$

In the same way we prove the result for

$$\underline{(0^{m+1}1)^i 0^m 1} \le a_1 < a_2 \le \underline{(0^{m+1}1)^{i-1}0^m 1}$$

two consecutive sequences in A_{p-1} for $i = p-3, p-2, \ldots, 2$.

We proceed in a similar way for the case ($\triangle$).

Therefore, we conclude that $\alpha \in A_\infty$ for any $\alpha \in \mathcal{A}_{n+1} \setminus \{\underline{0}, \underline{1}\}$. □

5. Proof of the main result

Let us now define $\mathcal{A} \subset \mathrm{Min}_2$ by $\mathcal{A} = \{a \in \mathrm{Min}_2 : \sigma(a) \ge \sigma(Sup\{\sigma^k(a) : k \in \mathbb{N}\})\}$. It is not hard to prove that $\mathcal{A}$ is a closed set.

Let us now prove the following

PROPOSITION (5.1). $\mathcal{A} = \overline{A_\infty}$.

Proof. To prove the inclusion $A_\infty \subset \mathcal{A}$ we proceed as in the proposition 3.3. That is: initially we prove that $A_0 \subset \mathcal{A}$, we assume, inductively, that $A_0 \cup \cdots \cup A_n \subset \mathcal{A}$ and we prove, exhaustively, that $(A_{n+1} \setminus A_n) \subset \mathcal{A}$. Hence, we conclude $A_\infty \subset \mathcal{A}$.

So, let us prove that $\mathcal{A} \subset \overline{A_\infty}$.

Let $a \in \mathcal{A}$ and $b(a) = \sup\{\sigma^i(a) : i \in \mathbb{N}\}$.

For $a = 0\underline{1}$ we have $\sigma(a) = \underline{1}$, $b(a) = \underline{1}$ and $a \in \mathcal{A}$. Since $a = \lim_{n\to\infty} \underline{01^n}$. We have that $a \in \overline{A_\infty}$.

For $a = \underline{0}$ we have $\sigma(a) = a$ and $b(a) = a$. Since $a = \lim_{n\to\infty} \underline{0^n 1}$. We have $a \in \overline{A_\infty}$.

For any other $a \in \mathcal{A}$ there is $n \in \mathbb{N}$ such that $\underline{0^{n+1}1} \le a \le \underline{0^n 1}$ or $\underline{01^n} \le a \le \underline{01^{n+1}}$.

Without loss of generality let us assume that $\underline{0^{n+1}1} \le a \le \underline{0^n 1}$.

For $a = \underline{0^n 1}$ we have $\sigma(a) = \underline{0^{n-1}10}$ and $b(a) = \underline{10^n}$. So, $\sigma(b(a)) = \underline{0^n 1} = a < \sigma(a)$.

That is $a \in \mathcal{A}$. The same is true for $a = \underline{0^{n+1}1}$.

Since, by definition, we have $\underline{0^{n+1}1} \in A_\infty$ and $\underline{0^n 1} \in A_\infty$ we have $a \in A_\infty$ in these cases.

Let us assume that $\underline{0^{n+1}1} < a < \underline{0^n1}$. It is clear that $b(a) \neq \underline{1}$. In fact, otherwise $\sigma(b(a)) = \underline{1} > \sigma(a)$ since $\underline{0^n1} < \sigma(a) < 0^{n-1}1\underline{0^n1}$. So, we must have $b(a) < \underline{1}$.

We note, in this case, that a cannot have two consecutive ones. In fact, for $a = 0^{n+1}11\ldots$ we have $\sigma(a) = 0^n11\ldots$. If $b(a) \geq 110^{s_1}\ldots$ then $\sigma(b(a)) \geq 10^{s_1} > \sigma(a)$, a contradiction.

For $a = 0^n10^{k_1}10^{k_2}1\cdots0^{k_s}11\cdots$ with $n \geq 2$ we also have obtain $\sigma(a) < \sigma(b(a))$, a contradiction. For the case $n = 1$ if $a = (01)^{r_1}1(01)^{r_2}1\cdots$ we have $a > \underline{01}$ a contradiction with our assumption $\underline{0^21} < a < \underline{01}$.

So, $a \in \mathcal{A}$ cannot have two consecutive ones and $a = 0^n(01)^{k_1}0^{k_2}(01)^{k_2}\cdots$.

Let us now assume that a is a periodic sequence

$$a = \underline{0^n(01)^{k_1}0^{k_2}(01)^{k_3}\cdots0^{k_{2r}}(01)^{k_{2r+1}}}.$$

(I) Assume $n \geq 2$ then we must have $k_1 = k_3 = \cdots = k_{2r+1} = 1$. Otherwise assume, for instance, that $k_1 \geq 2$. Then $b(a) \geq \underline{(10)^{k_1}0^{k_2-1}\cdots}$ and $\sigma(b(a)) \geq \underline{010\cdots} > \sigma(a) = 0^{n-1}(01)^{k_1}\cdots0^{k_{2r}}(01)^{k_{2r+1}}a$, a contradiction.

Hence, for

$$n \geq 2,\ a = \underline{0^n(01)0^{k_2}(01)\cdots0^{k_{2r}}01} = \underline{0^{n+1}10^{k_2+1}10^{k_4+1}1\cdots0^{k_{2r}+1}1}.$$

It is clear that $k_i \leq n$ because otherwise $a \notin \mathrm{Min}_2$, a contradiction. Moreover $k_{2i} \geq n-1$ for any $i = 1, 2, \ldots, r$. Otherwise assume k_{2j} defined by $k_{2j} = \min\{k_{2i} : i = 1, \ldots, r\}$ is such that $k_{2j} < n-1$. We must have $b(a) \geq \underline{10^{k_{2j}+1}10^{k_{2(j+1)}+1}\cdots0^{k_{2r}+1}10^{n+1}1\cdots10^{k_{2(j-1)}+1}}$ and $\sigma(b(a)) \geq 0^{k_{2j}+1}10^{k_{2(j+1)}+1}\cdots0^{k_{2r}}1a > \sigma(a) = 0^n1\cdots$ because $k_{2j}+1 < n$. Hence, $n-1 \leq k_{2i} \leq n$ for $i = 1, 2, \ldots, r$.

So, $a = \underline{0^n(01)0^{k_2}(01)\cdots0^{k_{2r}}(01)}$. Assume $k_{2r} = n$. In this case we must have $k_2 = k_4 = \cdots = k_{2r} = n$ and $a = \underline{0^{n+1}1} \in A_\infty$. If not, let us assume for instance that $k_2 = n-1$. In this situation we have $a = \underline{0^{n+1}10^n10^{k_4+1}10^{k_{2(r-1)}+1}10^{n+1}1}$ and for some j, $\sigma^j(a) = \underline{0^{n+1}10^{n+1}10^n1\cdots0^{k_{2(n-1)}}1} < a$. Therefore $k_{2r} = n-1$ and for the sequence a we get

$$\begin{aligned} a &= \underline{0^{n+1}10^{k_2+1}1\cdots0^{k_{2(r-1)}+1}10^n1} \\ &= \underline{0^n(01)0^{k_2}1\cdots0^{k_{2(n-1)}}(01)0^{n-1}(01)} \\ &= R_{0,01}(\underline{0^n10^{k_2}1\cdots0^{n-1}1}). \end{aligned}$$

Let $a_1 = \underline{0^n10^{k_2}1\cdots0^{n-1}1}$. It is clear that $a_1 \in \mathrm{Min}_2$. In fact, otherwise there exists j such that $k_{2j} = n$, $k_{2(j+1)} = k_2, \ldots, k_{2(j+p)} = k_{2p}$ and $k_{2(j+p+1)} > k_{2(p+1)}$. Applying these values to a we have that $a \notin \mathrm{Min}_2$, a contradiction. So, $a_1 \in \mathrm{Min}_2$.

It is clear that $b(a) = \underline{10^n10^{t_2+1}\cdots10^{t_{2r}+1}}$ if and only if

$$b(a_1) = \underline{10^{n-1}10^{t_2}\cdots10^{t_{2r}}}$$

and, consequently, $\sigma(b(a)) \leq b(a)$ imply $\sigma(b(a_1)) \leq \sigma(a_1)$. So, we conclude $a_1 \in \mathcal{A}$ and $\underline{0^n1} < a_1 < \underline{0^{n-1}1}$.

If $n-1 \geq 2$ we obtain $a_2 \in \mathcal{A}$ such that $a_1 = R_{0,01}(a_2)$ and $\underline{0^{n-1}1} < a_2 < \underline{0^{n-1}1}$. Successively, we will continue up to find $a_k \in \mathcal{A}$ such that $\underline{0^21} < a_k < \underline{01}$ and $a_{i-1} = R_{0,01}(a_i)$ for $i = 1, 2, \ldots, k$, $a_0 = a$.

(II) So, let us consider the case $\underline{0^21} < a < \underline{01}$ and $a \in \mathcal{A}$.

In this situation we must have

$$\begin{aligned} a &= \underline{(0^21)^{p_1}(01)^{p_2}(0^21)^{p_3}\cdots(0^21)^{p_{2r}-1}(01)^{p_{2r}}} \\ &= R_{0,01}(\underline{(01)^{p_1}1^{p_2}(01)^{p_3}\cdots(01)^{p_{2r}-1}1^{p_{2r}}}) \\ &= R_{01,1}\circ R_{01,1}(\underline{0^{p_1}1^{p_2}0^{p_3}\cdots 0^{p_{2r-1}}1^{p_r}}). \end{aligned}$$

As in the case (I) we can prove that $a_1 = \underline{0^{p_1}1^{p_2}\cdots 0^{p_{2r-1}}1^{p_{2r}}} \in \mathcal{A}$ and $\underline{0^{p_1}1} < a_1 < \underline{0^{p_1-1}1}$.

Now, we can apply the part (I) to a_1 and we have $a_1 = R_{0,01}(\tilde{a}_1)$ for some $\tilde{a}_1 \in \mathcal{A}$.

So, $a = R_{0,01}\circ R_{01,1}\circ R_{0,01}(\tilde{a}_1)$. Successively, we obtain that $a = R_{0,01}\circ R_{01,1}\circ R^{i_1}_{0,01}\circ R_{01,1}\circ R^{i_2}_{0,01}\circ\cdots\circ R^{i_k}_{0,01}(\underline{01})$. Hence, by proposition 3.3, we conclude that $a \in A_\infty$. Since any $a \in \mathcal{A}$ can be approximated by periodic sequences in $\mathcal{A}$ we conclude that $\mathcal{A} \subset \overline{A_\infty}$. □

To complete the proof of the Theorem (2.1) we need to prove the following

PROPOSITION (5.2). *For any $f \in \mathrm{DC}_c(2)$ the kneading sequence $a_f = I_f(0^+)$ satisfy $a \in \mathcal{A}$. The same result is true for $f \in \mathrm{DC}_c(1)$.*

Proof. Clearly $a \in \mathrm{Min}_2$. Since $f(0) = f(1)$ and $f|_{[0,c[}$ and $f|_{]c,1]}$ are increasing we have $f(x) \geq f(y)$ for any $x \in [0,c[$ and $y \in]c,1]$.

Assume $x_0 = f(0) = f(1) \in]0,1[$. Assume $\varepsilon > 0$ is such that $]x_0-\varepsilon, x_0+\varepsilon[\subset]0,1[$. Since $]x_0-\varepsilon, x_0] \subset f(]c,1])$ then there is $\delta > 0$ such that $f(1-\delta) = x_0-\varepsilon$ and for any $z \in]x_0-\varepsilon, x_0]$ there is $y \in]1-\delta, 1]$ such that $f(y) = z$.

Since $[x_0, x_0+\varepsilon[\subset f([0,c[)$ then there is $\tilde{\delta} > 0$ such that $f(\tilde{\delta}) = x_0+\varepsilon$ and for any $z \in]x_0, x_0+\varepsilon[$ there is $y \in [0,\tilde{\delta}[$ such that $f(y) = z$.

In this situation

$$I_f(x_0^+) = \lim_{z\downarrow x_0, z\in\Gamma_f} I_f(z) = \lim_{y\downarrow 0, y\in\Gamma_f} I_f(f(y)) = I_f(f(0^+)) = \sigma\circ I_f(0^+) = \sigma(a_f)$$

and

$$I_f(x_0^-) = \lim_{z\uparrow x_0, z\in\Gamma_f} I_f(z) = \lim_{y\uparrow 1, y\in\Gamma_f} I_f(f(y)) = I_f(f(1^-)) = \sigma\circ I_f(1^-) = \sigma(b_f).$$

Since $I_f(x_0^-) \leq I_f(x_0)$ we obtain $\sigma(b_f) \leq \sigma(a_f)$. For $b(a_f) = \sup\{\sigma^i(a_f) : i \in \mathbb{N}\}$ we have $b(a_f) \leq b_f$ and, consequently, $\sigma(b(a_f)) \leq \sigma(b_f)$. Hence, we obtain $\sigma(b(a_f)) \leq \sigma(a_f)$ as we claim. Clearly a similar argument apply for elements in $\mathrm{DC}_c(1)$. □

Theorem (2.1) is now a consequence of corollary (3.5) and proposition (5.2).

Remark. Now we can consider the sets $\mathrm{KDC}_c(b) = \{b_f : f \in DC(1)\cup \mathrm{DC}_c(2)\}$ and $B_\infty = \{b(a) : a \in A_\infty\}$ and, in a similar way as we did for the proof of the theorem 2.1, we can prove that $\mathrm{KDC}_c(b) = \overline{B_\infty}$.

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SPACE-LIKE HYPERSURFACES WITH CONSTANT k-TH MEAN CURVATURE IN $S_1^{n+1}(c)$

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ABSTRACT. In this paper, we give some characterizations of Riemannian product $H^m(c_1) \times S^{n-m}(c_2)$ and show that the Riemannian product $H^m(c_1) \times S^{n-m}(c_2)$ is the only complete connected space-like hypersurface in a de Sitter space $S_1^{n+1}(c)$ with constant k-th mean curvature $H_k > 0 (k < n)$ and two distinct principal curvatures, if (1) the multiplicities of both principal curvatures are greater than 1, in this case $1 < m < n-1$, or (2) one of the both principal curvatures is simple and $H_k^{2/k} < c$ and the sectional curvature of M^n is non-negative or the squared norm of the second fundamental form of M^n satisfies some pinching conditions, respectively, in this case $m = 1$ or $m = n-1$. We extend recent result of Z. Hu et al. [7].

1. Introduction

Let $S_1^{n+1}(c)$ be an $(n+1)$-dimensional de Sitter space with constant sectional curvature c $(c > 0)$. A hypersurface in a de Sitter space is said to be space-like if the induced metric on the hypersurface is positive definite.

In connection with the negative settlement of the Bernstein problem due to Calabi [3], Cheng-Yau [4] and Choquet-Bruhat et al. [5] proved for $c \geq 0$ and T. Ishihara [8] proved for $c < 0$ the following theorem.

THEOREM (1.1). ([4], [5], [8]). *Let M^n be an n-dimensional $(n \geq 2)$ complete maximal space-like hypersurface in an (n+1)-dimensional Lorentzian space form $M_1^{n+1}(c)$. Then*

(i) *if $c \geq 0$, M^n is totally geodesic.*

(ii) *if $c < 0$, then $S \leq n$ and $S = n$ if and only if $M^n = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$, where S denotes the squared norm of the second fundamental form of M^n.*

As a generalization of Theorem (1.1), complete space-like hypersurfaces with constant mean curvature in a Lorentz manifold have been investigated by many mathematicians. For example, let M^n be an n-complete space-like hypersurface with constant mean curvature in a de Sitter space $S_1^{n+1}(c)$, Goddard [6] conjectured that every such hypersurface must be totally umbilical. Akutagawa [2] and Ramanathan [14] had proved independently that Goddard's conjecture is true if $H^2 \leq c$ when $n = 2$, and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Montiel [12] solved this conjecture in the case when M is compact. Further

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discussions in this regard have been carried out by many other authors, we can see [7]-[11] and the author [15].

In [18] and [7], by considering the sectional curvature and the squared norm S of the second fundamental form of M^n, Zheng and Z. Hu et al. proved the following result, respectively.

THEOREM (1.2). ([18]). *Let M^n be an n-dimensional compact space-like hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(c)$ with constant scalar curvature $n(n-1)r$. If $r < c$ and the sectional curvature of M^n is non-negative, then M^n is isometric to a sphere.*

THEOREM (1.3). ([7]). *Let M^n be an n-dimensional $(n \geq 3)$ complete connected and oriented space-like hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple.*

(i) *If $r \neq 0$ and $S \geq (n-1)\frac{n(1-r)-2}{n-2} + \frac{n-2}{n(1-r)-2}$, then M^n is isometric to the hyperbolic cylinder $H^1(c_1)\times S^{n-1}(c_2)$, where $c_1 = -\frac{nr}{n(1-r)-2}$, $c_2 = \frac{nr}{n-2}$ and $r > 0$ or spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$, where $c_1 = \frac{nr}{n-2}$, $c_2 = -\frac{nr}{n(1-r)-2}$ and $r < 0$.*

(ii) *If $r > 0$ and $S \leq (n-1)\frac{n(1-r)-2}{n-2} + \frac{n-2}{n(1-r)-2}$, then M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$, where $c_1 = -\frac{nr}{n(1-r)-2}$ and $c_2 = \frac{nr}{n-2}$.*

We denote by h the second fundamental form of M^n and denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the principal curvatures at an arbitrary point of M^n. From [11], we know that the k-th mean curvature H_k of M^n is defined by

$$P_n(t) = (1+t\lambda_1)(1+t\lambda_2)\cdots(1+t\lambda_n) = 1 + C_n^1 H_1 t + \cdots + C_n^n H_n t^n,$$

that is, the k-th mean curvature H_k is the normalized k-th symmetric function of principal curvatures of the hypersurface M^n defined by

$$C_n^k H_k = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}, \tag{1.4}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

We should note that if $k=1$, H_1 is the mean curvature of M^n and if $k=2$, from (1.1) and (2.11), we have $H_2 = c - r$, where r is the normalized scalar curvature of M^n.

In this paper, we investigate complete hypersurfaces in a de Sitter space $S_1^{n+1}(c)$ with constant k-th mean curvature $H_k (k < n)$ and with two distinct principal curvatures. In order to state our theorem clearly, we introduce, see U-H.Ki et al. [10], the well-known standard models of complete space-like hypersurfaces with non-zero constant k-th mean curvature in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(c)$:

$$H^m(c_1)\times S^{n-m}(c_2) = \{(x,y) \in S_1^{n+1}(c) \subset R_1^{n+2} = R_1^{m+1}\times R^{n-m+1} : \ |x|^2 = -\frac{1}{c_1}, |y|^2 = \frac{1}{c_2}\},$$

where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}, c_1 < 0, c_2 > 0$ and $m = 1, \cdots, n-1$. We note that $H^m(c_1) \times S^{n-m}(c_2)$ in $S_1^{n+1}(c)$ has two distinct principal curvatures $\sqrt{c - c_1}$ with multiplicity m and $\sqrt{c - c_2}$ with multiplicity $n - m$.

From U-H. Ki et al. [10], $H^1(c_1) \times S^{n-1}(c_2)$ or $H^{n-1}(c_1) \times S^1(c_2)$ is, in particular, called a hyperbolic cylinder or a spherical cylinder in $S_1^{n+1}(c)$.

From above, we know that the hyperbolic cylinder or spherical cylinder has two distinct principal curvatures one of which is simple. Without loss of generality, we can denote the two distinct principal curvatures by λ and μ, and say that λ has multiplicity $n-1$ and μ has multiplicity 1. Therefore, from (1.1), we obtain

$$C_n^k H_k = C_{n-1}^k \lambda^k + C_{n-1}^{k-1} \lambda^{k-1} \mu,$$

this implies that

$$\lambda^{k-1}[(n-k)\lambda + k\mu] = nH_k. \tag{1.5}$$

For the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ and the spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$ in $S_1^{n+1}(c)$, we know that $\lambda \neq 0$, $\mu \neq 0$ and λ and μ satisfy

$$\lambda\mu = c. \tag{1.6}$$

From (1.2), we have

$$\mu = \frac{n}{k} H_k \lambda^{1-k} - \frac{n-k}{k} \lambda. \tag{1.7}$$

From (1.6) and (1.7), we know that λ satisfies

$$-\frac{n}{k} H_k \lambda^{2-k} + \frac{n-k}{k} \lambda^2 + c = 0,$$

that is,

$$ck\lambda^{k-2} + (n-k)\lambda^k - nH_k = 0.$$

Putting $t = \lambda^k$, we have

$$ckt^{\frac{k-2}{k}} + (n-k)t - nH_k = 0, \tag{1.8}$$

and the squared norm of the second fundamental form of the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or the spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$ is

$$S = (n-1)\lambda^2 + c^2\lambda^{-2} = (n-1)t^{2/k} + c^2 t^{-2/k},$$

where t satisfies (1.8).

Denote by $P_{H_k}(t)$ the following function

$$P_{H_k}(t) = ckt^{\frac{k-2}{k}} + (n-k)t - nH_k, \quad (t > 0, \quad H_k > 0), \tag{1.9}$$

where $c > 0$ and $H_k^{2/k} < c$. From Lemma (3.21), we know that (1.9) has a positive real root t_1.

We shall prove the following result:

MAIN THEOREM. *Let M^n be an n-dimensional ($n \geq 3$) complete connected space-like hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(c)$ with constant k-th mean curvature $H_k > 0 (k < n)$ and with two distinct principal curvatures. Then*

(1) *if the multiplicities of both principal curvatures are greater than 1, then* M^n *is isometric to the Riemannian product* $H^m(c_1)\times S^{n-m}(c_2)$, *where* $\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}$, $c_1<0$, $c_2>0$ *and* $1<m<n-1$.

(2) *if one of the both principal curvatures is simple and* $H_k^{2/k}<c$, *then* M^n *is isometric to the hyperbolic cylinder* $H^1(c_1)\times S^{n-1}(c_2)$ *or spherical cylinder* $H^{n-1}(c_1)\times S^1(c_2)$, $\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}$, $c_1<0$, $c_2>0$, *if one of the following conditions is satisfied:*

(i) *the sectional curvature of* M^n *is non-negative on* M^n, *or*

(ii) $S\le(n-1)t_1^{2/k}+c^2t_1^{-2/k}$ *on* M^n, *where* $k>2$ *or*

(iii) $S\ge(n-1)t_1^{2/k}+c^2t_1^{-2/k}$ *on* M^n, *where* $k>2$ *and* t_1 *is the positive real root of* (1.9).

Remark (1.10). If $c=1$, $k=1$ and $k=2$, the result of (1) in Main Theorem was proved by A. Brasil Jr et al. [9] and Z. Hu et al. [7], respectively.

Remark (1.11). We know that if $c=1$ and $k=2$, Z. Hu et al. [7] obtained an interesting result, see Theorem (1.3). In this paper, we extend recent result of Z. Hu et al. [7] to the case $k>2$.

2. Preliminaries

Let M^n be an n-dimensional space-like hypersurface in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(c)$. We choose a local field of semi-Riemannian orthonormal frames $\{e_1,\dots,e_{n+1}\}$ in $S_1^{n+1}(c)$ such that at each point of M^n, $\{e_1,\dots,e_n\}$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1\le A,B,C,\dots\le n+1;\quad 1\le i,j,k,\dots\le n.$$

Let $\{\omega_1,\dots,\omega_{n+1}\}$ be the dual frame field so that the semi-Riemannian metric of $S_1^{n+1}(c)$ is given by $d\bar{s}^2=\sum_i\omega_i^2-\omega_{n+1}^2=\sum_A\epsilon_A\omega_A^2$, where $\epsilon_i=1$ and $\epsilon_{n+1}=-1$.

The structure equations of $S_1^{n+1}(c)$ are given by

$$d\omega_A=\sum_B\epsilon_B\omega_{AB}\wedge\omega_B,\quad \omega_{AB}+\omega_{BA}=0, \tag{2.1}$$

$$d\omega_{AB}=\sum_C\epsilon_C\omega_{AC}\wedge\omega_{CB}+\Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB}=-\frac{1}{2}\sum_{C,D}K_{ABCD}\omega_C\wedge\omega_D, \tag{2.3}$$

$$K_{ABCD}=\epsilon_A\epsilon_Bc(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}). \tag{2.4}$$

Restrict these forms to M^n, we have

$$\omega_{n+1}=0. \tag{2.5}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.8}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.9}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{2.10}$$

is the second fundamental form of M^n.

From the above equation, we have

$$n(n-1)(r-c) = S - n^2H^2, \tag{2.11}$$

where $n(n-1)r$ is the scalar curvature of M^n, H is the mean curvature, and $S = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form of M^n.

We choose $e_1, \cdots, e_n$ such that $h_{ij} = \lambda_i\delta_{ij}$. From (2.6) we have

$$\omega_{n+1i} = \lambda_i\omega_i, \quad i = 1, 2, \cdots, n. \tag{2.12}$$

Hence, we have from the structure equations of M^n

$$\begin{aligned} d\omega_{n+1i} &= d\lambda_i \wedge \omega_i + \lambda_i d\omega_i \\ &= d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j. \end{aligned} \tag{2.13}$$

On the other hand, we have for the curvature forms of $S_1^{n+1}(c)$,

$$\begin{aligned} \Omega_{n+1i} &= -\tfrac{1}{2}\sum_{C,D} K_{n+1iCD}\omega_C \wedge \omega_D \\ &= \tfrac{1}{2}\sum_{C,D} c(\delta_{n+1C}\delta_{iD} - \delta_{n+1D}\delta_{iC})\omega_C \wedge \omega_D \\ &= c\omega_{n+1} \wedge \omega_i = 0. \end{aligned} \tag{2.14}$$

Therefore, from the structure equations of $S_1^{n+1}(c)$, we have

$$\begin{aligned} d\omega_{n+1i} &= \sum_j \omega_{n+1j} \wedge \omega_{ji} - \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i} \\ &= \sum_j \lambda_j\omega_{ij} \wedge \omega_j. \end{aligned} \tag{2.15}$$

From (2.13) and (2.15), we obtain

$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j)\omega_{ij} \wedge \omega_j = 0. \tag{2.16}$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij}. \tag{2.17}$$

Then $\psi_{ij} = \psi_{ji}$. (2.16) can be written as

$$\sum_j (\psi_{ij} + \delta_{ij} d\lambda_j) \wedge \omega_j = 0. \tag{2.18}$$

By E. Cartan's Lemma, we get

$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k Q_{ijk}\omega_k, \tag{2.19}$$

where Q_{ijk} are uniquely determined functions such that $Q_{ijk} = Q_{ikj}$.

On the other hand, since the covariant derivative of the second fundamental form h_{ij} of M^n is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{kj}\omega_{ki},$$

from $h_{ij} = \lambda_i \delta_{ij}$, we have

$$\sum_k h_{ijk}\omega_k = \delta_{ji} d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}.$$

Combining with (2.19), we know that $Q_{ijk} = h_{ijk}$. From (2.6) and the Codazzi equations $h_{ijk} = h_{ikj}$, we have $h_{ijk} = h_{jik} = h_{ikj}$, that is

$$Q_{ijk} = Q_{jik} = Q_{ikj}. \tag{2.20}$$

3. Proof of Main Theorem

We firstly state a Proposition which can be proved by making use of the similar method due to Otsuki [13] for Riemannian space forms (see [7]).

PROPOSITION (3.1). *Let M^n be a hypersurface in an (n+1)-dimensional de Sitter space $S_1^{n+1}(c)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Proof of (1) in Main Theorem. Let λ and μ be the two distinct principal curvatures of multiplicities m and $n-m$ respectively, where $1 < m < n-1$. From (1.4), we have

$$C_n^k H_k = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k},$$

this is always a equality of H_k, λ and μ, we can denote it by

$$C_n^k H_k = \mathcal{F}(\lambda, \mu). \tag{3.2}$$

Denote by D_λ and D_μ the integral submanifolds of the corresponding distribution of the space of principal vectors corresponding to the principal curvature λ and μ, respectively. From Proposition (3.1), we know that λ is constant on D_λ. Since the k-th mean curvature H_k is constant, (3.2) implies that μ is constant on D_λ. By making use of Proposition (3.1) again, we have μ is constant on D_μ. Therefore, we know that μ is constant on M^n. By the same assertion we know that λ is constant on M^n. Therefore M^n is isoparametric. By the congruence Theorem of Abe, Koike and Yamaguchi [1], we know that M^n is isometric to the Riemannian product $H^m(c_1) \times S^{n-m}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$ and $1 < m < n-1$. This completes the proof of (1) in Main Theorem.

From now on, we consider that $n(n \geq 3)$-dimensional complete connected space-like hypersurface with constant k-th mean curvature $H_k > 0$ $(k < n)$ and with two distinct principal curvatures, one of which is simple. Without loss of generality, we may assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu,$$

where λ_i for $i = 1, 2, \cdots, n$ are the principal curvatures of M^n. Therefore, we obtain

$$C_n^k H_k = C_{n-1}^k \lambda^k + C_{n-1}^{k-1} \lambda^{k-1} \mu,$$

this implies that

$$\lambda^{k-1}[(n-k)\lambda + k\mu] = nH_k. \tag{3.3}$$

For $k \geq 2$, if $\lambda = 0$ at some point, from (3.2), we have $H_k = 0$ at this point, this is a contradiction to the fact that $H_k > 0$. By changing the orientation for M^n and renumbering $e_1, \ldots, e_n$ if necessary, we may assume that $\lambda > 0$. Therefore, we have for all k

$$\mu = \frac{n}{k} H_k \lambda^{1-k} - \frac{n-k}{k}\lambda. \tag{3.4}$$

Since

$$\lambda - \mu = n\frac{\lambda^k - H_k}{k\lambda^{k-1}} \neq 0,$$

we know that $\lambda^k - H_k \neq 0$.

Let $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$. We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k}\, \omega_k, \quad d\mu = \sum_{k=1}^{n} \mu_{,k}\, \omega_k. \tag{3.5}$$

From Proposition (3.1), we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \ \text{ on } \ M_1^{n-1}(x). \tag{3.6}$$

From (3.4), we have

$$d\mu = [\frac{n(1-k)}{k} H_k \lambda^{-k} - \frac{n-k}{k}] d\lambda. \tag{3.7}$$

Thus, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \ \text{ on } \ M_1^{n-1}(x). \tag{3.8}$$

In this case, we may consider locally that λ is a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.19) and (3.6), we have for $1 \le j \le n-1$,

$$\begin{aligned} d\lambda = d\lambda_j &= \sum_{k=1}^{n} Q_{jjk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n = \lambda,_n \omega_n. \end{aligned} \tag{3.9}$$

Therefore, we have

$$Q_{jjk} = 0, \quad 1 \le k \le n-1, \quad \text{and} \quad Q_{jjn} = \lambda,_n\,. \tag{3.10}$$

By (2.19) and (3.8), we have

$$\begin{aligned} d\mu = d\lambda_n &= \sum_{k=1}^{n} Q_{nnk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n = \sum_{i=1}^{n} \mu,_i\,\omega_i = \mu,_n\,\omega_n,\,. \end{aligned} \tag{3.11}$$

Hence, we obtain

$$Q_{nnk} = 0, \quad 1 \le k \le n-1, \quad \text{and} \quad Q_{nnn} = \mu,_n\,. \tag{3.12}$$

From (3.7), we get

$$Q_{nnn} = \mu,_n = [\frac{n(1-k)}{k} H_k \lambda^{-k} - \frac{n-k}{k}]\lambda,_n\,. \tag{3.13}$$

From the definition of ψ_{ij}, if $i \ne j$, we have $\psi_{ij} = 0$ for $1 \le i \le n-1$ and $1 \le j \le n-1$. Therefore, from (2.19), if $i \ne j$ and $1 \le i \le n-1$ and $1 \le j \le n-1$ we have

$$Q_{ijk} = 0, \quad \text{for any} \quad k. \tag{3.14}$$

By (2.19), (2.20), (3.10), (3.12), (3.13) and (3.14), we get

$$\begin{aligned} \psi_{jn} &= \sum_{k=1}^{n} Q_{jnk}\omega_k \\ &= Q_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda,_n\,\omega_j. \end{aligned} \tag{3.15}$$

From (2.19), (3.4) and (3.15) we have

$$\begin{aligned} \omega_{jn} &= \frac{\psi_{jn}}{\lambda - \mu} = \frac{\lambda,_n}{\lambda - \mu}\omega_j \\ &= \frac{k\lambda^{k-1}\lambda,_n}{n(\lambda^k - H_k)}\omega_j. \end{aligned} \tag{3.16}$$

Therefore, from the structure equations of M^n we have

$$d\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0.$$

Therefore, we may put $\omega_n = ds$. By (3.9) and (3.11), we get

$$d\lambda = \lambda,_n\, ds, \quad \lambda,_n = \frac{d\lambda}{ds},$$

and

$$d\mu = \mu,_n\, ds, \qquad \mu,_n = \frac{d\mu}{ds}\,.$$

Then we have

$$\omega_{jn} = \frac{k\lambda^{k-1}\lambda,_n}{n(\lambda^k - H_k)}\omega_j = \frac{k\lambda^{k-1}\frac{d\lambda}{ds}}{n(\lambda^k - H_k)}\omega_j = \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\omega_j. \tag{3.17}$$

From (3.17) and the structure equations of $S_1^{n+1}(c)$, we have

$$\begin{aligned} d\omega_{jn} &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn}\wedge\omega_{nn} - \omega_{jn+1}\wedge\omega_{n+1n} + \Omega_{jn} \\ &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} - \omega_{jn+1}\wedge\omega_{n+1n} - c\omega_j\wedge\omega_n \\ &= \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k - (c-\lambda\mu)\omega_j\wedge ds. \end{aligned}$$

From (3.17), we have

$$\begin{aligned} d\omega_{jn} &= \frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}d\omega_j \\ &= \frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k \\ &= \{-\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} + [\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}]^2\}\omega_j\wedge ds \\ &+ \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k. \end{aligned}$$

From the above two equalities, we have

$$\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} - \{\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\}^2 - (c-\lambda\mu) = 0. \tag{3.18}$$

From (3.4) we get

$$\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} - \{\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\}^2 + \frac{n}{k}H_k\lambda^{2-k} - \frac{n-k}{k}\lambda^2 - c = 0. \tag{3.19}$$

Since we define $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$, we obtain from the above equation

$$\frac{d^2\varpi}{ds^2} + \varpi\frac{ck\lambda^{k-2} + (n-k)\lambda^k - nH_k}{k\lambda^{k-2}} = 0. \tag{3.20}$$

We can prove the following Lemmas:

LEMMA (3.21). *Let*

$$P_{H_k}(t) = ckt^{\frac{k-2}{k}} + (n-k)t - nH_k, \quad (t > 0, \quad H_k > 0, \quad k > 2).$$

where $c > 0$ *and* $H_k^{2/k} < c$. *Then* $P_{H_k}(t)$ *has a positive real root* t_1 *and*

(i) *if* $t > H_k$, *then* $P_{H_k}(t) > 0$;

(ii) *if* $t < H_k$, *then* $t \geq t_1$ *holds if and only if* $P_{H_k}(t) \geq 0$ *and* $t \leq t_1$ *holds if and only if* $P_{H_k}(t) \leq 0$.

Proof. We have

$$\frac{dP_{H_k}(t)}{dt} = c(k-2)t^{-2/k} + (n-k). \tag{3.22}$$

For $k > 2$, we easily see that

$$\frac{dP_{H_k}(t)}{dt} > 0,$$

it follows that $P_{H_k}(t)$ is a strictly monotone increasing function of t and $\lim_{t\to+\infty} P_{H_k}(t) = +\infty$. For $k > 2$, we have $P_{H_k}(0) = -nH_k < 0$, Therefore, from the continuity property of $P_{H_k}(t)$, we infer that $P_{H_k}(t)$ has a positive real root, which can be denoted by t_1.

Now we prove the second part of Lemma (3.21).

Since $k > 2$, we know that $P_{H_k}(t)$ is a strictly monotone increasing function of t and has a positive real root t_1. From $H_k^{2/k} < c$, we have $P_{H_k}(H_k) = kH_k^{\frac{k-2}{k}}(c - H_k^{2/k}) > 0$. Thus, we have $H_k > t_1$. Therefore, from the strictly monotone increasing property of $P_{H_k}(t)$, we know that (i) if $t > H_k$, then $P_{H_k}(t) > P_{H_k}(H_k) > 0$, (ii) if $t < H_k$, then $t \geq t_1$ holds if and only if $P_{H_k}(t) \geq P_{H_k}(t_1) = 0$ and $t \leq t_1$ holds if and only if $P_{H_k}(t) \leq P_{H_k}(t_1) = 0$. We complete the proof of Lemma (3.21). □

LEMMA (3.23). *Let*

$$S(t) = \frac{1}{k^2 t^{(2k-2)/k}}\{(n-1)k^2t^2 + [nH_k - (n-k)t]^2\}, \quad (t > 0, \quad H_k > 0, \quad k > 2),$$

where $c > 0$ *and* $H_k^{2/k} < c$. *If* $t \leq H_k$, *then* $t \geq t_1$ *holds if and only if* $S(t) \leq (n-1)t_1^{2/k} + c^2 t_1^{-2/k}$ *and* $t \leq t_1$ *holds if and only if* $S(t) \geq (n-1)t_1^{2/k} + c^2 t_1^{-2/k}$, *where* t_1 *is the positive real root of (1.9).*

Proof. We have

$$\frac{dS(t)}{dt} = \frac{2t^{(2-3k)/k}}{k^3}\{(n^2 - 2nk + nk^2)t^2 + n(k-2)(n-k)H_k t + (1-k)n^2H_k^2\},$$

it follows that the solution of $\frac{dS(t)}{dt} = 0$ is $t = H_k$. Therefore, we know that $t \leq H_k$ holds if and only if $S(t)$ is a decreasing function, $t \geq H_k$ holds if and only if $S(t)$ is an increasing function and $S(t)$ obtain its minimum at $t = H_k$ (also see [17]).

Since $t \leq H_k$ if and only if $S(t)$ is a decreasing function, we infer that if $t \leq H_k$, then $t \geq t_1$ holds if and only if

$$\begin{aligned} S(t) \leq S(t_1) &= \frac{1}{k^2 t_1^{(2k-2)/k}}\{(n-1)k^2t_1^2 + [nH_k - (n-k)t_1]^2\} \\ &= \frac{1}{k^2 t_1^{(2k-2)/k}}\{(n-1)k^2t_1^2 + [nH_k - (n-k)t_1 - ckt_1^{\frac{k-2}{k}} + ckt_1^{\frac{k-2}{k}}]^2\} \\ &= \frac{1}{k^2 t_1^{(2k-2)/k}}\{(n-1)k^2t_1^2 + [-P_{H_k}(t_1) + ckt_1^{\frac{k-2}{k}}]^2\} \\ &= (n-1)t_1^{2/k} + c^2 t_1^{-2/k}, \end{aligned}$$

and $t \leq t_1$ holds if and only if $S(t) \geq S(t_1) = (n-1)t_1^{2/k} + c^2t_1^{-2/k}$. We complete the proof of Lemma (3.23). □

LEMMA (3.24). *If $H_k^{2/k} < c$, then the positive function ϖ is bounded.*

Proof. From the definition of ϖ and (3.20), we have

$$\begin{aligned} \frac{d^2\varpi}{ds^2} &+ \varpi[-\frac{n}{k}H_k(H_k + \varpi^{-n})^{\frac{2}{k}-1} \\ &+ \frac{n-k}{k}(H_k + \varpi^{-n})^{\frac{2}{k}} + c] = 0, \quad \text{for } \lambda^k - H_k > 0, \end{aligned} \tag{3.25}$$

or

$$\begin{aligned} \frac{d^2\varpi}{ds^2} &+ \varpi[-\frac{n}{k}H_k(H_k - \varpi^{-n})^{\frac{2}{k}-1} \\ &+ \frac{n-k}{k}(H_k - \varpi^{-n})^{\frac{2}{k}} + c] = 0, \quad \text{for } \lambda^k - H_k < 0. \end{aligned} \tag{3.26}$$

By making use of the following integral formula

$$\int u^m(a+bu^q)^p du = \frac{u^{m+1}(a+bu^q)^p}{pq+m+1} + \frac{apq}{pq+m+1}\int u^m(a+bu^q)^{p-1}du,$$

where all m, p, q, a, b are not zero and all m, p, q are rational number, we have

$$\begin{aligned} -\frac{n-k}{k}\int \varpi(H_k \pm \varpi^{-n})^{\frac{2}{k}}d\varpi = \frac{1}{2}\varpi^2(H_k \pm \varpi^{-n})^{\frac{2}{k}} \\ - \frac{n}{k}H_k\int \varpi(H_k \pm \varpi^{-n})^{\frac{2}{k}-1}d\varpi. \end{aligned}$$

Integrating (3.25) or (3.26), we can get

$$(\frac{d\varpi}{ds})^2 + c\varpi^2 - \varpi^2(H_k + \varpi^{-n})^{\frac{2}{k}} = C, \quad \text{for } \lambda^k - H_k > 0,$$

or

$$(\frac{d\varpi}{ds})^2 + c\varpi^2 - \varpi^2(H_k - \varpi^{-n})^{\frac{2}{k}} = C, \quad \text{for } \lambda^k - H_k < 0,$$

where C is a constant. Therefore, we have

$$\varpi^2[c - (H_k + \varpi^{-n})^{\frac{2}{k}}] \leq C, \quad \text{for } \lambda^k - H_k > 0, \tag{3.27}$$

or

$$\varpi^2[c - (H_k - \varpi^{-n})^{\frac{2}{k}}] \leq C, \quad \text{for } \lambda^k - H_k < 0. \tag{3.28}$$

Since we assume that $H_k^{2/k} < c$, we have $c - H_k^{2/k} > 0$, from (3.27) and (3.28), we infer that the positive function ϖ is bounded from above. We complete the proof of Lemma (3.24). □

Proof of (2) in Main Theorem. (i) From (3.18), we have

$$\frac{d^2\varpi}{ds^2} + \varpi(c - \lambda\mu) = 0\,. \tag{3.29}$$

If the sectional curvature of M^n is non-negative, that is, for $i \neq j$, $R_{ijij} = c - \lambda_i\lambda_j \geq 0$, we have $c - \lambda\mu \geq 0$. From (3.29), we have $\frac{d^2\varpi}{ds^2} \leq 0$. Thus, $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, by the similar assertion in Wei [16], we have $\varpi(s)$ must be monotonic when s tends to infinity. Since we assume that $H_k^{2/k} < c$, from Lemma (3.24), we know that the positive function $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty} \varpi(s)$ and $\lim_{s\to+\infty} \varpi(s)$ exist and then we get

$$\lim_{s\to-\infty} \frac{d\varpi(s)}{ds} = \lim_{s\to+\infty} \frac{d\varpi(s)}{ds} = 0. \tag{3.30}$$

From the monotonicity of $\frac{d\varpi(s)}{ds}$, we have $\frac{d\varpi(s)}{ds} \equiv 0$ and $\varpi(s) = constant$. From $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$ and (3.3), we have λ and μ are constant, that is, M^n is isoparametric. Therefore, by the congruence Theorem of Abe, Koike and Yamaguchi [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$.

(ii) If $S \leq (n-1)t_1^{2/k} + c^2 t_1^{-2/k}$, putting $t = \lambda^k$, we obtain that $S = S(t)$. From (3.20), we have

$$\frac{d^2\varpi}{ds^2} + \varpi\frac{1}{kt^{(k-2)/k}}P_{H_k}(t) = 0. \tag{3.31}$$

Since

$$\lambda - \mu = n\frac{\lambda^k - H_k}{k\lambda^{k-1}} \neq 0,$$

we know that $\lambda^k - H_k \neq 0$. Thus, we have $t \neq H_k$. We consider two cases $t > H_k$ or $t < H_k$.

If $t > H_k$, from Lemma (3.21), we know that $P_{H_k}(t) > 0$. From (3.31), we have $\frac{d^2\varpi(s)}{ds^2} < 0$. This implies that $\frac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s and thus it has at most one zero point for $s \in (-\infty, +\infty)$. If $\frac{d\varpi(s)}{ds}$ has no zero point in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in $(-\infty, +\infty)$. If $\frac{d\varpi(s)}{ds}$ has exactly one zero point s_0 in $(-\infty, +\infty)$, then $\varpi(s)$ is a monotone function of s in both $(-\infty, s_0]$ and $[s_0, +\infty)$.

On the other hand, from Lemma (3.24), we know that $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s\to-\infty} \varpi(s)$ and $\lim_{s\to+\infty} \varpi(s)$ exist and (3.30) holds. This is impossible because $\frac{d\varpi(s)}{ds}$ is a strictly monotone decreasing function of s. Therefore, we know that the case $t > H_k$ does not occur.

If $t < H_k$, from Lemma (3.21), Lemma (3.23) and (3.31), we have $S(t) \leq (n-1)t_1^{2/k} + c^2 t_1^{-2/k} = S(t_1)$ holds if and only if $t \geq t_1$ if and only if $P_{H_k}(t) \geq 0$ and if and only if $\frac{d^2\varpi}{ds^2} \leq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Since we assume that $H_k^{2/k} < c$, from Lemma (3.24), we know that the positive function $\varpi(s)$ is bounded. By the same assertion in the proof of (i) in (2), we know that λ and μ must be constant, that is, M^n is isoparametric. By the congruence Theorem of Abe, Koike and Yamaguchi [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$.

(iii) If $S \geq (n-1)t_1^{2/k} + c^2 t_1^{-2/k}$, by the similar assertion in the case (ii), we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$. This completes the proof of (2) in Main Theorem. □

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