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SOCIEDAD MATEMATICA MEXICANA

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SOME UPPER BOUNDS FOR COMPOSITION NUMBERS

PAUL Y. YIU

Abstract

The composition number $r *_{\mathbb{Z}} s$ is the length of the shortest expression of the polynomial $(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2)$ as a sum of squares of integral bilinear forms. For $r, s \leq 16$, these composition numbers have been completely determined in [Y2] and [Y3]. The present paper continues the study in [SY] on upper bounds of composition numbers in the range $r, s \leq 32$. Several new formulae, notably those of types [15,30,63], [18,27,57], and [40,14,64], are constructed in the form of consistently signed intercalate matrices, leading to improvements over [SY]. Upper bounds are also given for $r *_{\mathbb{Z}} r$, $r = 33, \ldots, 64$, and for $r *_{\mathbb{Z}} s$, $33 \leq r \leq 64$, $10 \leq s \leq 16$.

1. Introduction

The construction of polynomial identities of the form

$$(1) \qquad (x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2$$

with *integer* coefficients has been studied since very early times. We shall refer to an identity like (1) above as a *composition formula* of type $[r, s, n]_{\mathbb{Z}}$, or simply $[r, s, n]$. Such formulae are combinatorial in nature, in that one of type $[r, s, n]$ is equivalent to a *consistently signed intercalate matrix* of type $[r, s, n]$. We recall the definition from [Y1]. Let M be an $r \times s$ matrix with generic entry $M(i; j)$, which we shall think of as a *color*.

Definitions. (a) An *intercalate matrix* of type (r, s, n) matrix is an $r \times s$ with n distinct colors satisfying the following conditions.

(i) The colors along each row or column are distinct.

(ii) If $M(i; j) = M(i'; j')$, then $M(i; j') = M(i'; j)$.

(b) An intercalate matrix M can be signed consistently if it is possible to endow each entry $M(i; j)$ with a sign $\epsilon_{i,j} = \pm 1$ such that

$$\epsilon_{i,j}\epsilon_{i,j'}\epsilon_{i',j}\epsilon_{i',j'} = -1 \quad \text{whenever } M(i; j) = M(i', j') \text{ for } i \neq i',\ j \neq j'.$$

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A METHOD TO GENERATE UPPER BOUNDS FOR THE SUMS OF SQUARES FORMULAE PROBLEM

ADOLFO SÁNCHEZ-FLORES

Abstract

The number $r *_Z s$ is the smallest integer t such that there exists a formula of the form $(x_1^2+\cdots+x_r^2)(y_1^2+\cdots+y_s^2) = z_1^2+\cdots+z_t^2$, where each z_i is a bilinear form in the sets of indeterminates X and Y with integer coefficients. It is well known that finding a formula of this type is equivalent to obtaining an $r \times s$ *consistently signed intercalate matrix* of type $[r, s, t]$. Most of the best upper bounds known for $r *_Z s$ $(1 \le r, s \le 64)$ are obtained by juxtaposing two of these matrices of smaller size. However, an "irreducible" juxtaposition of 5 matrices was recently given, providing the best upper bounds known for some $r *_Z s$. Here we describe all the irreducible configurations that juxtapose k matrices, for $k \le 9$. Also, we show an algorithm that, for given r, $s \ge 1$, and a set C of upper bounds on $i *_Z j$ $(i \le r, j \le s$, and $i+j < r+s)$, produces the juxtaposition of $k \le 7$ matrices that yields the lowest bound on $r *_Z s$ induced by C. Finally, with the best upper bounds known to date, we use this algorithm to determine new upper bounds for $r *_Z s$ $(1 \le r, s \le 64)$.

1. Introduction

An old and difficult problem is to find, given integers r and s and a commutative ring Λ of characteristic not 2, the smallest integer t (generally denoted by $r *_\Lambda s$) such that there exists a *product formula of size* (r, s, t) *over the ring* Λ [5,6], that is, an identity of the form

$$(x_1^2+\cdots+x_r^2)(y_1^2+\ldots+y_s^2) = z_1^2+\ldots+z_t^2,$$

where each z_i is a bilinear form in the sets of indeterminates X and Y with coefficients in Λ. Besides its intrinsic interest, this problem has been extensively studied since it appears in several branches of mathematics (see [9,10] for an account).

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A NOTE ON A THEOREM OF D. MUMFORD

GEORGE KEMPF

Let $1 \to \mathbf{G}_m \to G \to H \to 0$ be an extension of an additive abelian group scheme H by $\mathbf{G}_m$ in the center. Then $g_1 g_2 g_1^{-1} g_2^{-1}$ is in $\mathbf{G}_m$. So we get a pairing $G \times G \to \mathbf{G}_m$ which is bicommulative and skew-symetric. As it is trivial on $\mathbf{G}_m \times \mathbf{G}$ and $G \times G_m$. It comes to a pairing $(\ ,\ )$, $H \times H \to G_m$.

G is a theta group if $\mathbf{G}_m$ is the center of G; equivalently the pairing $(\ ,\ )$ is non-degenerate.

THEOREM. *If G is a theta group, G has a unique irreducible representation where $\mathbf{G}_m$ acts by multiplication. This representation has dimension =* $\sqrt{\deg H}$.

Let $\bar{A}$ be a maximal abelian closed subgroup of G containing $\mathbf{G}_m$. Then $\bar{A}$ is the inverse image of a closed subgroup A of H. We may find a closed subgroup $\bar{\bar{A}}$ of $\bar{A}$ which is isomorphic to A. Clearly A is the dual abelian group of H/A under pairing on which G_m acts by multiplication.

LEMMA. *Let M be a non-zero representation of G. Then M contains non-zero vector v which is fixed by $\bar{A}$.*

A. s A is abelian, M has a non-eigenvector w with eigenvalue $\chi: A \to \mathbf{G}_m$. Let g be an element of G such that $g^{-1}ag = a\chi^{-1}(a)$ for all a in $\bar{\bar{A}}$. Then $g \cdot w$ is fixed by $\bar{\bar{A}}$.

Let v be a vector fixed by $\bar{\bar{A}}$. Then we have the morphism $G \to M$ sending g to $g \cdot v$. Thus we get a cohomorphism $M^{\wedge} \to \Gamma(G, Q_G)_1 = R$ where the $_1$ means right eigenvectors for the character $\bar{A} \to \mathbf{G}_m$ which is the identity on $\mathbf{G}_m$ and trivial on $\bar{\bar{A}}$. Now R is a left comodule of G. Hence we have a homomorphism of G representing $\psi: R^V \to M$.

Claim. ψ is an isomorphism if M is irreducible. We want to show that ψ is injective. If its kernel is non-zero, it contains a non-zero invariant but this is impossible by the next result because the invariant evaluation at $\bar{A}$ goes to a non-zero equivariant in M.

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CHOW VARIETIES OF ABELIAN VARIETIES

E. JAVIER ELIZONDO AND RICHARD M. HAIN

Abstract

We prove that if A is an abelian variety over $\mathbb{C}$ acting algebraically on a complex projective variety X, then the Euler characteristic of X equals the Euler characteristic of the fixed point set X^A. We obtain that if A is an abelian variety and X is a principal A-bundle over a projective variety Y, then the Euler characteristic of a Chow variety in X equals either zero or the Euler characteristic of a Chow variety of Y.

1. Introduction

Let X be a projective variety, and let λ be an element of the homology group $H = H_{2p}(X, \mathbb{Z})$ of X modulo torsion. The restricted Chow variety $\mathscr{C}_\lambda$, of X with respect λ, is the projective variety of all effective cycles on X with homology class λ. Very little is known about these varieties. Their Euler characteristics were computed in [2] when X is a complete toric variety.

The purpose of this paper is twofold. First, to give a new proof of a Theorem of Lawson and Yau [4]; our version is Theorem (4.1). Secondly, to apply this result to compute the Euler characteristic of $\mathscr{C}_\lambda$ in some particular cases.

More precisely, Theorem (4.1) states that if an abelian variety A, defined over $\mathbb{C}$, acts algebraically on a complex projective variety X, then $\chi(X) = \chi(X^A)$, where $\chi(X)$ is the (topological) Euler characteristic of X and $\chi(X^A)$ the Euler characteristic of the fixed point set X^A.

The relevance of the result is illustrated by Corollary (4.2): If A is an abelian variety and X is a principal A-bundle over a projective variety Y, then

$$\chi\left(\mathscr{C}_\lambda(X)\right) = \begin{cases} \chi\left(\mathscr{C}_\beta(Y)\right) & \text{if } \alpha = \pi^*(\beta) \text{ for some } \beta \in H_\bullet(Y); \\ 0 & \text{otherwise.} \end{cases}$$

This is obtained by applying Theorem (4.1) to the action of A on $\mathscr{C}_\lambda(X)$.

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VARIETIES AND INDUCTION

JON F. CARLSON

Abstract

Suppose that H is a subgroup of a finite group G and that k is an algebraically closed field of characteristic $p > 0$. Let V be a closed subvariety of the maximal ideal spectrum of the cohomology ring $H^*(G, k)$. We give necessary and sufficient conditions on V and H which imply that any kG-module whose variety is contained in V must be stable induced from a kH-module. The results also apply to infinitely generated modules. For an application we look at the vertices of modules, defined over Chevalley groups in the defining characteristic, whose varieties are supported on root subgroups.

1. Introduction

In the last few years Dave Benson, Jeremy Rickard and the author have been involved in a project to extend the notions and methods of support varieties for finitely generated modules over group algebras to the category of all modules. For a finitely generated module, the definition of its variety is reasonably straightforward. If G is a finite group, k a field of characteristic p and M a finitely generated kG-module, then the variety of M is the closed set in the maximal ideal spectrum of $H^*(G, k)$ corresponding to the ideal which is the annihilator of $\mathrm{Ext}^*_{kG}(M, M)$. For an infinitely generated module things are much more complicated. For one thing the "variety" of an infinitely generated module is not really a variety but rather is a subset of the prime ideal spectrum of the cohomology ring.

The project was originally motivated by an attempt to recover the Krull-Schmidt Theorem in complexity quotient categories [3]. It has also led to the discovery of several unexpected results. In one exciting paper Rickard [9] has shown that the stable category of all kG-modules modulo projectives has idempotent objects which correspond to thick subcategories defined by varieties. These objects have played a role in the development of an extended definition of cohomological varieties for modules and an extension of the Tensor Product Theorem to the category of all kG-modules [4].

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IRREDUCIBLE COMPONENTS AND ISOLATED POINTS IN THE BRANCH LOCUS OF THE MODULI SPACE OF SMOOTH CURVES

Esteban Gómez González

Abstract

In this paper we give a description of the isolated points in the branch locus $\mathscr{B}_g$ of the moduli space of smooth curves of genus greater than or equal to three and we study their automorphism group. To obtain this description, we present a rigorous construction of the irreducible components of $\mathscr{B}_g$ as calculated by Cornalba.

0. Introduction

Let $\mathscr{M}_g$ be the moduli space of smooth curves of genus $g \geq 3$ over an algebraically closed field of characteristic zero. The branch locus of $\mathscr{M}_g$ is defined as:

$$\mathscr{B}_g = \Big\{[X] \in \mathscr{M}_g \ : \ \mathrm{Aut}(X) \neq \{\mathrm{Id}\}\Big\}$$

It is known that for $g \geq 4$, $\mathscr{B}_g$ coincides with the singular locus of $\mathscr{M}_g$ ([7]). This definition is not valid for genus 2 because all the curves are hyperelliptic; $\mathscr{B}_2$ is defined as the locus of curves of genus 2 whose group of automorphisms is different from $\mathbb{Z}/2\mathbb{Z}$.

In this paper, we show a characterization of the isolated points in $\mathscr{B}_g$ ($g \geq 3$): A point $[X] \in \mathscr{M}_g$ is an isolated point in $\mathscr{B}_g$ if and only if $2g+1$ is a prime number and the group of automorphisms of the curve X that represents this point is cyclic of order $2g+1$ (Theorem (3.5)). Moreover, we deduce another characterization in terms of cyclic coverings of prime order of the projective line with three branch points (Theorem (3.7)), and for each genus we compute the number of isolated points and the equations of the singular plane curves corresponding to these points.

To prove these characterizations, we strongly use the irreducible components of $\mathscr{B}_g$ calculated by Cornalba in [1]. For this reason, we exhibit a detailed proof of the existence, irreducibility and dimension of such components

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INTERSECTION DYNAMICS ON GRASSMANN MANIFOLDS

Ernesto Rosales-González

Abstract

Let $G(k, m)$ be the space of k planes in the m-dimensional complex space $\mathbb{C}^m$. The aim of this work is to study how the images of one point in $G(k, m)$ under the iteration of an automorphism of $G(k, m)$ fall on the differents stratums of a stratified submanifold Y in $G(k, m)$. In particular, if the subspace Y is the union $Y = Y_0 \cup \ldots \cup Y_k$, where Y_i is the set of k planes in $\mathbb{C}^m$ whose intersection with a $m - k$-dimensional fixed subspace have dimension i and the automorphism A is induced by a linear one in $\mathbb{C}^m$, then the N iteration $A^N(X)$ of one point X falls on each strata Y_i periodically with respect to N when N is sufficiently large.

1. Introduction

Let M be an m-dimensional smooth manifold. One problem in dynamical systems consists in the study of the behavior of the asymptotic growth of the numbers associated to the intersection in M of one submanifold Y with the images of another submanifold X under the iterates of a smooth map A from M to itself.

For example, define in the cartesian product $M = L \times L$ (L a smooth manifold) the diagonal $\Delta = \{(x, x) : x \in L\}$ as the submanifolds X and Y. The graph of a map $g : L \to L$ is the image of the diagonal under the map $A = (Id, g)$. So the intersection of $A^N(X)$ and Y consists of those points (x, x) on the diagonal where x is a fixed point of g^N. In their work [1], M. Artin and B. Mazur studied the asymptotical growth of the number of the isolated points of the intersection $A^N(X) \cap Y$ and proved that in generic cases the growth of the number of isolated fixed points of the map g^N is at most exponential for any g.

In the more general situation we define X, Y as smooth submanifolds of a compact smooth manifold M and $A : M \to M$ is a k-differentiable map. For this case, Arnold in his work [2] considered the number $|A^N(X) \cap Y|$ where $|\ \ |$ denotes the d-measure and $d = \dim X + \dim Y - \dim M$. For the particular case when the dimension of X and Y are complementary, and the number $|A^N(X) \cap$

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THE SPREADING MODELS OF THE SPACE $\mathscr{J} = (J \oplus J \oplus \dots)_{l^2}$

HELGA FETTER AND BERTA GAMBOA DE BUEN

Abstract

We study the bounded sequences in $\mathscr{J}$ and show that the spreading models of $\mathscr{J}$ are isomorphic either to the James space J or to l^2.

1. Introduction

The James space J was introduced in 1950 by R.C. James [5] and is of fundamental importance for the study of the geometry of Banach spaces, specially as a source of counterexamples; the results concerning this space used here can be found in [4].

We define the James space J by the set of all real sequences $x = (a_1, a_2, \dots)$ such that $\lim_n a_n = 0$ for which

$$\|x\| = \sup\left(\frac{1}{2}\sum_{i=0}^{n}(a_{p_{i+1}} - a_{p_i})^2\right)^{\frac{1}{2}} < \infty,$$

where the sup is taken over all choices of n and all choices of positive integers $0 = p_0 < p_1 < \dots < p_{n+1}$ and $a_0 = 0$.

A. Brunel and L. Sucheston, while studying the convergence of certain series in Banach spaces, introduced the notion of spreading models of a Banach space X in [3]; for a detailed account of these spaces the reader is referred to the book by B. Beauzamy and J.T. Lapresté [2].

It was shown by Andrew [1] that if X is a spreading model of J, then X is either isomorphic to l^2 or X is isomorphic to J. We will see that the same result holds for $\mathscr{J}$.

For this we will first fix some notation and give some known results necessary for our proof.

In J there are two main bases, namely the canonical basis $\{e_n\}_n$ which is monotone and shrinking and the summing basis $\{\xi_n\}_n$ where $\xi_n = e_1 +$

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