Uniform controllability of PDEs with vanishing coefficients by complex-analytic methods

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Abstract

We consider the problem of the estimation of the cost of null controllability of a fourth order parabolic partial differential equation when the coefficients of the higher order terms vanish. The corresponding problem for second order equations has been considered in several works: For the heat equation with vanishing viscosity coefficient, we can cite [2], [5], where are used Carleman estimates and energy estimates. These results were improved in [6], [7], [4], and [3], using different techniques, mainly moment method.

Concerning fourth order equations, as far we known, the only reference is [1], where it is studied a parabolic equation posed in $(0, L) \times (0, T)$, and composed by a transport term with velocity M and a fourth order term with vanishing viscosity. It is proved that, under the hypothesis $T \geq 40L/|M|$, the cost of controllabiity remains uniformly bounded with respect to the diffusion coefficient. The main tools are Carleman estimates and diffusion estimates.

The problem which motivated this work is to improve the previous result, by using the complex variable approach introduced in [4].

More precisely, given $\varepsilon > 0$, we consider the system given by

$$\begin{cases} y_t + \varepsilon y_{xxxx} + \delta y_{xxx} + M y_x = 0, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0, \ y(t,L) = 0, & t \in (0,T), \\ By(t,0) = h(t), \ By(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{1}$$

where $\delta = -2\varepsilon^{2/3} M^{1/3}$ and $By = 2\varepsilon y_{xx} + \delta y_x$.

Our main result is the following.

Theorem 1 *Given $L, T > 0$, there exist $c, C > 0$ s.t. $\forall\ y^0 \in L^2(0,L)$, $\varepsilon \in (0,1)$, there is $u \in L^2(0,T)$ s.t. the solution y satisfies*

$$y(\cdot, T) = 0 \in L^2(0,L), \quad \|u\|_{L^2(0,T)} \leq C \exp(-c/\varepsilon^{1/3}) \left\|y^0\right\|_{L^2(0,L)},$$

whenever

$$T > 5L/M,\ M > 0; \quad T > 7L/|M|,\ M < 0.$$

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References

[1] Nicolás Carreño and Patricio Guzmán, *On the cost of null controllability of a fourth-order parabolic equation*, J. Differential Equations **261** (2016), no. 11, 6485–6520. MR 3552571

[2] J.-M. Coron and S. Guerrero, *Singular optimal control: a linear 1-D parabolic-hyperbolic example*, Asymptot. Anal. **44** (2005), no. 3-4, 237–257. MR 2176274

[3] Jérémi Dardé and Sylvain Ervedoza, *On the cost of observability in small times for the one-dimensional heat equation*, (2017).

[4] Olivier Glass, *A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit*, J. Funct. Anal. **258** (2010), no. 3, 852–868. MR 2558179

[5] S. Guerrero and G. Lebeau, *Singular optimal control for a transport-diffusion equation*, Comm. Partial Differential Equations **32** (2007), no. 10-12, 1813–1836. MR 2372489

[6] Pierre Lissy, *Explicit lower bounds for the cost of fast controls for some 1-D parabolic or dispersive equations, and a new lower bound concerning the uniform controllability of the 1-D transport-diffusion equation*, J. Differential Equations **259** (2015), no. 10, 5331–5352. MR 3377528

[7] ______, *The cost of the control in the case of a minimal time of control: the example of the one-dimensional heat equation*, J. Math. Anal. Appl. **451** (2017), no. 1, 497–507. MR 3619248