Control, Observation and Identification Problems for the Wave Equation on Metric Graphs

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Quantum graphs are metric graphs with differential equations defined on the edges. Recent interest in control and inverse problems for quantum graphs is motivated by applications to important problems of classical and quantum physics, chemistry, biology, and engineering. For trees, i.e. graphs without cycles, various types of inverse and control problems were discussed in the literature (see, e.g. [1],[2] and references therein) and almost nothing (with several exceptions [3],[4],[5]) was done for graphs with cycles. In this talk we describe exact controllability and identifiability results for the wave and Schrödinger equations on general compact graphs.

Let $\Gamma = \{E, V\}$ be a finite compact connected metric graph, where $E = \{e_j\}_{j=1}^N$ is a set of edges and $V = \{v_j\}_{j=1}^M$ is a set of vertices. We recall that a graph is called a *metric graph* if every edge $e_j \in E$ is identified with an interval of the real line with a positive length. Let the degree of a vertex, $deg(v)$, be the number of edges incident to it and $\partial\Gamma$ be the set of boundary vertices, i.e. $\partial\Gamma = \{v \in V \,|\, deg(v) = 1\}$.

The metric graph Γ determines naturally the Hilbert space of square integrable functions $\mathcal{H} = L^2(\Gamma) = \{\phi = (\phi_j)_{j=1}^N,\ \phi_j \in L^2(e_j)\}$. We define the space $\mathcal{H}^1$ of continuous functions ϕ on Γ such that $\phi_j \in H^1(e_j)$ for every $e_j \in E$ and $\phi|_{\partial\Gamma} = 0$.

Let $q = (q_j)_{j=1}^N$ be a real valued function such that $q_j \in L^\infty(e_j)$. We consider the initial boundary value problem for the wave equation on the graph:

$$u_{tt} - u_{xx} + q(x)u = 0 \text{ in } \{\Gamma \setminus V\} \times (0,T) \tag{1}$$

$$\begin{array}{l} \sum_{e_j \in E(v)} \partial u_j(v,t) = 0 \ \text{ at each vertex } v \in V \setminus \partial\Gamma, \text{ and } t \in [0,T] \\ u(\cdot,t) \text{ is continuous at each vertex, for } t \in [0,T] \end{array} \tag{2}$$

$$u = 0 \text{ on } \partial\Gamma \times [0,T], \quad u|_{t=0} = u_0 \in \mathcal{H}^1,\ u_t|_{t=0} = u_1 \in \mathcal{H} \text{ in } \Gamma. \tag{3}$$

In (**??**) and below $\partial u_j(v,t)$ denotes the derivative of u at the vertex v taken along the edge e_j in the direction outwards the vertex and $E(v)$ is the set of all edges incident to v.

We prove that a proper choice of observations in the form of directional derivatives of u at some boundary and internal vertices guarantees observability of the system. More exactly, we specify subsets $V^* \subset V$ and $E^*(v) \subset E(v)$ such that the following statement is true.

Theorem 1. *For sufficiently large T there are exist positive constants c and C such that for any $(u_0, u_1) \in \mathcal{H}^1 \times \mathcal{H}$, the solution u of the system (1)–(3) satisfies the (observability) inequalities*

$$c\ \left[\|u_0\|^2_{\mathcal{H}^1} + \|u_1\|^2_{\mathcal{H}}\right] \leq \int_0^T \left[\sum_{v \in V^*} \sum_{e_j \in E^*(v)} |\partial u_j(v,t)|^2\right] dt \leq C\ \left[\|u_0\|^2_{\mathcal{H}^1} + \|u_1\|^2_{\mathcal{H}}\right].$$

We describe the observability sets (of the vertices V^* and directions E^*) and give estimates of the observability time T. The corresponding exact controllability result can be formulated using the standard duality argument.

The inverse problem consists of reconstructing the graph's topology, the lengths of all edges and the functions q_j on them from the observations (directional derivatives of u) described in Theorem 1. The problem is solved using a new version of the leaf peeling method proposed for trees in [2]. We prove the uniqueness theorem and describe the constructive identification algorithm.

References

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