Control of Systems of Stefan Type

Miroslav Krstic

Partial differential equations with moving/free boundaries have been principally used to describe the time evolution of the boundaries themselves in a variety of problems of physical interest. The canonical model with a moving boundary is the "Stefan problem", whose original late-1890s purpose was the modeling of thermodynamics of phase changes such as melting from solid phase to liquid phase, or solidification from liquid phase to solid phase. Typical applications of contemporary interest which involve phase change include sea ice melting and freezing, additive manufacturing for materials of both polymer and metal, cancer treatment via cryosurgeries, and lithium-ion batteries. Apart from the thermodynamical model, the moving boundary problems of Stefan type has been employed to model several chemical, electrical, and social dynamics such as tumor growth processes, domain walls in ferroelectric thin films, spreading of invasive species in ecology, and information diffusion on social networks.

The mathematical description of the Stefan problem is given by a diffusion PDE for a temperature profile defined on a time-varying spatial domain. The dynamics of the domain's boundary are governed by an ODE that is driven by the Neumann boundary value of the PDE's state at the moving boundary itself. Such an evolution of the spatial domain, which is not know a priori in time, makes the entire coupled PDE-ODE system nonlinear, in spite of the linearity of the PDE in the domain's interior, and makes the (infinite) dynamic order of the system nonconstant in time, due to the nonconstancy of the domain size/volume. Those qualities, in turn, make the Stefan problem quite challenging for mathematical analysis, let alone control design.

In this talk, I review a few results from the doctoral work of Shumon Koga on boundary control design for the Stefan problem in one dimensional spatial coordinate. Motivated by manufacturing process, the control objective is set to drive the interface position between the liquid phase and the solid phase to a desired setpoint by manipulating a boundary heat flux at the liquid phase to promote the melting process. First, we consider the so called "one-phase" Stefan problem in which the temperature profile of the solid phase is assumed to be the equillibrium melting temperature. The proposed control design via backstepping method satisfies a positivity condition on the temperature, which ensures that the dynamics remain within the model's physical validity (no islands of solid emerge within the liquid phase) and achieves the exponential stabilization of the temperature profile and the moving interface globally (for all positive initial conditions) to the desired setpoint in the H1 norm. Next, we discuss several extensions, to the extent that the time permits: the ”two-phase” Stefan problem in which the dynamics of the temperature profiles in both the liquid and solid phase are incorporated, observer design and output feedback for the Stefan problem, and the design for the Stefan problem which, in addition to diffusion, contains convection.

This presentation is based on several publications, including [S. Koga, M. Diagne, and M. Krstic, “Control and state estimation of the one-phase Stefan problem via backstepping design,” IEEE Transactions on Automatic Control, 2019].