On the reachable space of systems governed by parabolic equations in one space dimension

Andreas Hartmann*, Karim Kellay†, Marius Tucsnak‡

We consider the system

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \frac{\partial}{\partial x}\left[a(x)\dfrac{\partial w}{\partial x}(t,x)\right] & t \geqslant 0,\ x \in (0,\pi), \\ w(t,0) = u_0(t), \quad w\,(t,\pi) = u_\pi(t) & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\pi)\,, \end{cases} \tag{0.1}$$

where a is a positive function on $[0,\pi]$ which can be extended to a holomorphic function on $\mathbb{C}$. The above systems models the heat propagation in a rod of length π and conductivity $a(\cdot)$, controlled by prescribing the temperature at both ends. It is well known (see, for instance, [5, Proposition 10.7.3]) that for every $u_0,\ u_\pi \in L^2[0,\infty)$ the problem (0.1) admits a unique solution $w \in C([0,\infty), W^{-1,2}(0,\pi))$. (Recall that $W^{-1,2}(0,\pi)$ is the dual of the usual Sobolev space $W_0^{1,2}(0,\pi)$ with respect to the pivot space $L^2[0,\pi]$.) Moreover, according to the same reference, the *input-to-state maps* $(\Phi_\tau)_{\tau>0}$ defined by

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = w(\tau,\cdot) \qquad (\tau > 0,\ u_0,\ u_\pi \in L^2[0,\tau]), \tag{0.2}$$

lie, for every $\tau > 0$, in $\mathcal{L}(L^2([0,\tau];\mathbb{C}), W^{-1,2}(0,\pi))$.

This work aims providing a precise description of the *reachable space*, which is the space of states which can be attained at instant τ when the input is freely moving in $L^2([0,\tau];U)$. In our case this space is the range of Φ_τ, denoted $\operatorname{Ran}\Phi_\tau$, where Φ_τ has been defined in (0.2). As far as we know, the first results on this space for the boundary controlled heat equation ($a \equiv 1$) have been given in the classical paper of Fattorini and Russell [2], where it is shown that the functions which extend holomorphically to a horizontal strip containing $[0,\pi]$ and vanishing, together with all their derivatives of even order, at $x = 0$ and $x = \pi$, belong to $\operatorname{Ran}\Phi_\tau$. This result implies, in particular, that $\operatorname{Ran}\Phi_\tau \supset \operatorname{Ran}\mathbb{T}_\tau$, where $\mathbb{T}$ is the semigroup generated by the $1D$ Dirichlet Laplacian in $W^{-1,2}(0,\pi)$, which means that the system determined by (0.1) is *null-controllable* in any time $\tau > 0$.

The fact that some other types of functions (like polynomials), not necessarily vanishing at the extremities of the considered interval, are in the reachable space has been remarked in a

*Institut de Mathématiques de Bordeaux, Université de Bordeaux/Bordeaux INP/CNRS , 351 Cours de la Libération, 33 405 TALENCE, France, andreas.hartman@u-bordeaux.fr

†Institut de Mathématiques de Bordeaux, Université de Bordeaux/Bordeaux INP/CNRS, 351 Cours de la Libération, 33 405 TALENCE, France, karim.kellay@u-bordeaux.fr

‡Institut de Mathématiques de Bordeaux UMR 5251, Université de Bordeaux/Bordeaux INP/CNRS, 351 Cours de la Libération, 33 405 TALENCE, France, marius.tucsnak@u-bordeaux.fr

series of papers published in the eighties (see, for instance, Schmidt [4] and references therein). More recently, in Martin, Rosier and Rouchon [3], it has been shown that any function which can be extended to a holomorphic map in a disk centered in $\frac{\pi}{2}$ and of diameter $\pi e^{(2e)^{-1}}$ lies in the reachable space. This result has been further improved in Dardé and Ervedoza [1], where it has been shown that any function which can be extended to one which is holomorphic in a neighbourhood of the square D defined by

$$D = \{s = x + iy \in \mathbb{C} \ \mid \ |y| < x \ \text{ and } \ |y| < \pi - x\}, \tag{0.3}$$

lies in the reachable space.

With the above definitions, the result in [3] (when restricted to inputs in $L^2[0,\tau]$), which asserts that $\operatorname{Ran}\Phi_\tau \subset \operatorname{Hol}(D)$ (the space formed by the functions holomorphic on D), can be strengthened (still for $a \equiv 1$) to:

Proposition 0.1. *For every $\tau > 0$ we have $\operatorname{Ran}\Phi_\tau \subset A^2(D)$, where the* Bergman space *$A^2(D)$ consists of all functions holomorphic in D with $\int_D |f(x+iy)|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty$..*

Our main result improves the existing lower bounds of the reachable space and states as follows:

Theorem 0.2. *Assume that $a \equiv 1$. then For every $\tau > 0$ we have $\operatorname{Ran}\Phi_\tau \supset E^2(D)$, where the Hardy-Smirnov space $E^2(D)$ is given by*

$$E^2(D) = \left\{ f \in \mathrm{HOL}(D) \ \mid \ \int_{\partial D} |f(\zeta)|^2 \, |\mathrm{d}\zeta| < \infty \right\}.$$

To conclude, we discuss the case of a variable conductivity a and some perspectives for nonlinear equations.

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